ADDITIONAL MATHEMATICS

KEY.POINTS

Exam Guide

O LEVEL

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NEW SOLUTIONS. NEW MINDS.

REDSPOT
PUBLISHING

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OVERVIEW OF THE TOPIC

SIMULTANEOUS EQUATIONS *2 key areas*	2 SIMULTANEOUS LINEAR EQUATIONS	• Elimination method
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2 SIMULTANEOUS LINEAR EQUATIONS

Objective: Solve 2 simultaneous linear equations using either elimination or substitution method

☑ Simultaneous equations are equations that have common unknowns and therefore needed to be solved together.

☑ 2 simultaneous linear equations are of the form:

2 simultaneous linear equations	$a_1 y + b_1 x = c_1$ $a_2 y + b_2 x = c_2$	where x and y are common unknowns and a_1, a_2, b_1, b_2, c_1, c_2 are constants

☑ To solve 2 simultaneous linear equations, the method commonly used is elimination method (although substitution method can also be used):

① Elimination Method	step ❶ Eliminate one unknown first by equating (multiply or divide) and cancelling out (add or substract) an identical term step ❷ Solve the equation for the remaining unknown step ❸ Substitute the solution obtained in ❷ into any one of the equations to solve for the eliminated unknown (usually 1 solution to each unknown)

STOP AND THINK

[Q] Solve the following simultaneous equations:

$$3x - 4y = 11$$
$$7x - 5y = 4$$

SOLUTION

$3x - 4y = 11 \cdots\cdots\cdots (1)$
$7x - 5y = 4 \cdots\cdots\cdots (2)$

$(1)\times 5$: $15x - 20y = 55 \cdots\cdots\cdots (3)$
$(2)\times 4$: $28x - 20y = 16 \cdots\cdots\cdots (4)$

$(4)-(3)$: $(28x - 20y) - (15x - 20y) = 16 - 55$
$\Rightarrow 28x - 15x = -39$
$\Rightarrow 13x = -39$
$\therefore x = -3$

Substitute $x = -3$ into (1): $3(-3) - 4y = 11$
$\Rightarrow -9 - 4y = 11$
$\Rightarrow -4y = 20$
$\therefore y = -5$

$\therefore x = -3,\ y = -5$ **(Ans)**

NON-LINEAR EQUATIONS

Objective: Identify non-linear equations and use substitution method to solve the equations

☑ Any equation not in the form of $\boxed{ay + bx = c}$ is considered non-linear equation.

☑ When we deal with non-linear equations, there can be more than one solution. Elimination method may not able to help us obtain the answer.

☑ A pair of simultaneous equations, one linear and one non-linear, or both non-linear, are usually solved using substitution method:

② Substitution Method	step ❶ Express one of the unknowns in terms of the other unknown in one of the equations to form a new expression (use linear equation first, if there is one) step ❷ Substitute this new expression into the unused equation and solve for the unknown step ❸ Substitute the solution(s) obtained in ❷ into the new expression to solve for the other unknown

EXAM TIP

When we solve 2 simultaneous equations, the solutions are actually the intersection points (x and y coordinates) of the 2 intersecting curves (or straight lines) represented by the simultaneous equations.

STOP AND THINK

[Q] Solve the following simultaneous equations:

$$3x+4y=2$$
$$x^2+8xy+12=0$$

SOLUTION

$$3x+4y=2 \quad \cdots\cdots\cdots \ (1)$$
$$x^2+8xy+12=0 \quad \cdots\cdots\cdots \ (2)$$

From (1): $3x+4y=2$

$$\Rightarrow \ 3x=2-4y$$
$$\Rightarrow \ x=\frac{2-4y}{3} \quad \cdots\cdots\cdots \ (3)$$

Substitute (3) into (2):

$$x^2+8xy+12=0$$
$$\Rightarrow \quad \left(\frac{2-4y}{3}\right)^2+8\left(\frac{2-4y}{3}\right)y+12=0$$
$$\Rightarrow \quad \frac{4-16y+16y^2}{9}+\frac{16y-32y^2}{3}+12=0$$
$$\times 9: \Rightarrow \quad 4-16y+16y^2+3\left(16y-32y^2\right)+(9)(12)=0$$
$$\Rightarrow \quad 4-16y+16y^2+48y-96y^2+108=0$$
$$\Rightarrow \quad -80y^2+32y+112=0$$
$$\Rightarrow \quad 5y^2-2y-7=0$$
$$\Rightarrow \quad (5y-7)(y+1)=0$$

$$\Rightarrow \quad 5y-7=0 \quad \text{or} \quad y+1=0$$
$$\therefore \ y=\frac{7}{5} \qquad \therefore \ y=-1$$

Substitute $y=\frac{7}{5}$ into (3): $x=\dfrac{2-4\left(\frac{7}{5}\right)}{3}$

$$=-\frac{18}{5}\times\frac{1}{3}$$
$$=-\frac{6}{5}$$
$$=-1\frac{1}{5}$$

Substitute $y=-1$ into (3): $x=\dfrac{2-4(-1)}{3}$

$$=\frac{6}{3}$$
$$=2$$

$\therefore \ x=-1\frac{1}{5},\ y=1\frac{2}{5}$ or $x=2,\ y=-1$ **(Ans)**

SAMPLE QUESTIONS

1. Solve the following simultaneous equations:

   (a) $x + 2y = 8$
   $3x + y = 9$

   (b) $11x + 2y = 10$
   $9x - 7y = 0$

   (c) $x^2 + y^2 = 13$
   $3x + y = 9$

   (d) $x(3 + y) = 30$
   $y(2 + 2x) = 28$

   (e) $\frac{2}{x} + \frac{3}{y} = 13$
   $\frac{1}{xy} = 6$

   (f) $3y = 5 - x$
   $\frac{x}{y} + \frac{8y}{x} = 6$

2. Given that $(a, 7)$ is a solution of the simultaneous equations $3x - y = 8$ and $bx^2 - xy + 9 = y^2$, find

   (a) the value of a and b,

   (b) the coordinates of the other solution.

3. Given that the difference between 2 positive numbers is 5 and the sum of their squares is 377, find the numbers.

4. Find the coordinates of the points of intersection of the line $x + y = 3$ and the curve $x^2 - 2x + 2y^2 = 3$.

5. The line $y = 2x - 3$ meets the curve $3x^2 - y^2 - 3x - 5y + 6 = 0$ at A and B. Calculate the coordinates of A and B.

6. The simultaneous equations $ax + by = 8$ and $a^2x - b^2y = -10$ have a solution $x = 2$, $y = 3$. Find the values of the constants a and b.

WORKED SOLUTIONS

1. (a) $x+2y=8 \cdots\cdots\cdots (1)$
$3x+y=9 \cdots\cdots\cdots (2)$

$(2)\times 2$: $6x+2y=18 \cdots\cdots\cdots (3)$

$(3)-(1)$: $5x=10$

$\therefore\ x=2$

Substitute $x=2$ into (1): $2+2y=8$

$\Rightarrow\ 2y=6$

$\therefore\ y=3$

$\therefore\ x=2,\ y=3$ **(Ans)**

(b) $11x+2y=10 \cdots\cdots\cdots (1)$
$9x-7y=0 \cdots\cdots\cdots (2)$

$(1)\times 7$: $77x+14y=70 \cdots\cdots\cdots (3)$

$(2)\times 2$: $18x-14y=0 \cdots\cdots\cdots (4)$

$(3)+(4)$: $95x=70$

$\therefore\ x=\frac{70}{95}$

$=\frac{14}{19}$

Substitute $x=\frac{14}{19}$ into (1): $11\left(\frac{14}{19}\right)+2y=10$

$\Rightarrow\ 2y=10-\frac{154}{19}$

$\therefore\ y=5-\frac{77}{19}$

$=\frac{18}{19}$

$\therefore\ x=\frac{14}{19},\ y=\frac{18}{19}$ **(Ans)**

(c) $x^2+y^2=13 \cdots\cdots\cdots (1)$
$3x+y=9 \cdots\cdots\cdots (2)$

From (2): $3x+y=9$

$\Rightarrow\ y=9-3x \cdots\cdots\cdots (3)$

Substitute (3) into (1):

$x^2+(9-3x)^2=13$

$\Rightarrow\ x^2+81-54x+9x^2=13$

$\Rightarrow\ 10x^2-54x+68=0$

$\Rightarrow\ 5x^2-27x+34=0$

$\Rightarrow\ (5x-17)(x-2)=0$

$\Rightarrow\ 5x-17=0$ or $x-2=0$

$x=\frac{17}{5}$ $\qquad x=2$

Substitute $x=2$ into (3): $y=9-3(2)$

$=3$

Substitute $x=\frac{17}{5}$ into (3): $y=9-3\left(\frac{17}{5}\right)$

$=-\frac{6}{5}$

$=-1\frac{1}{5}$

$\therefore\ x=2,\ y=3$ or $x=3\frac{2}{5},\ y=-1\frac{1}{5}$ **(Ans)**

(d) $x(3+y)=30 \cdots\cdots\cdots (1)$
$y(2+2x)=28 \cdots\cdots\cdots (2)$

From (1): $x=\frac{30}{3+y} \cdots\cdots\cdots (3)$

Substitute (3) into (2):

$y\left(2+2\times\frac{30}{3+y}\right)=28$

$\Rightarrow\ 2y+\frac{60y}{3+y}=28$

$\Rightarrow\ 2y(3+y)+60y=28(3+y)$

$\Rightarrow\ 6y+2y^2+60y=84+28y$

$\Rightarrow\ 2y^2+38y-84=0$

$\Rightarrow\ y^2+19y-42=0$

$\Rightarrow\ (y+21)(y-2)=0$

$\Rightarrow\ y+21=0$ or $y-2=0$

$y=-21$ $\qquad y=2$

Substitute $y=-21$ into (3): $x=\frac{30}{3+(-21)}$

$=-\frac{30}{18}$

$=-1\frac{2}{3}$

Substitute $y=2$ into (3): $x=\frac{30}{3+(2)}$

$=6$

$\therefore\ x=-1\frac{2}{3},\ y=-21$ or $x=6,\ y=2$ **(Ans)**

(e) $\frac{2}{x}+\frac{3}{y}=13 \cdots\cdots\cdots (1)$
$\frac{1}{xy}=6 \cdots\cdots\cdots (2)$

From (2): $y=\frac{1}{6x} \cdots\cdots\cdots (3)$

Substitute (3) into (1):

$\frac{2}{x}+3(6x)=13$

$\Rightarrow\ 2+18x^2=13x$

$\Rightarrow\ 18x^2-13x+2=0$

$\Rightarrow\ (9x-2)(2x-1)=0$

$\Rightarrow\ 9x-2=0$ or $2x-1=0$

$x=\frac{2}{9}$ $\qquad x=\frac{1}{2}$

Substitute $x=\frac{2}{9}$ into (3): $y=\frac{1}{6}\left(\frac{9}{2}\right)$

$=\frac{3}{4}$

Substitute $x=\frac{1}{2}$ into (3): $y=\frac{1}{6}(2)$

$=\frac{1}{3}$

$\therefore\ x=\frac{2}{9},\ y=\frac{3}{4}$ or $x=\frac{1}{2},\ y=\frac{1}{3}$ **(Ans)**

(f) $3y = 5 - x$ .... (1)

$\frac{x}{y} + \frac{8y}{x} = 6$ .... (2)

From (1): $x = 5 - 3y$ .... (3)

Substitute (3) into (2):

$$\frac{5-3y}{y} + \frac{8y}{5-3y} = 6$$

$\Rightarrow \quad (5-3y)^2 + 8y^2 = 6y(5-3y)$

$\Rightarrow \quad 25 - 30y + 9y^2 + 8y^2 = 30y - 18y^2$

$\Rightarrow \quad 35y^2 - 60y + 25 = 0$

$\Rightarrow \quad 7y^2 - 12y + 5 = 0$

$\Rightarrow \quad (7y-5)(y-1) = 0$

$\Rightarrow \quad 7y - 5 = 0$ or $y - 1 = 0$

$y = \frac{5}{7}$ $\qquad y = 1$

Substitute $y = \frac{5}{7}$ into (3): $x = 5 - 3\left(\frac{5}{7}\right)$

$= \frac{20}{7}$

$= 2\frac{6}{7}$

Substitute $y = 1$ into (3): $x = 5 - 3(1)$

$= 2$

$\therefore$ $x = 2\frac{6}{7}$, $y = \frac{5}{7}$ or $x = 2$, $y = 1$ **(Ans)**

2. (a) $3x - y = 8$ .... (1)

$bx^2 - xy + 9 = y^2$ .... (2)

Since $(a, 7)$ is a solution to the equations,

From (1): $3a - 7 = 8$

$\Rightarrow 3a = 15$

$\therefore a = 5$ **(Ans)**

From (2): $b(5)^2 - (5)(7) + 9 = (7)^2$

$\Rightarrow 25b - 35 + 9 = 49$

$\Rightarrow 25b = 75$

$\therefore b = 3$ **(Ans)**

(b) $3x - y = 8$ .... (1)

$3x^2 - xy + 9 = y^2$ .... (2)

From (1): $y = 3x - 8$ .... (3)

Substitute (3) into (2):

$3x^2 - x(3x-8) + 9 = (3x-8)^2$

$\Rightarrow \quad 3x^2 - 3x^2 + 8x + 9 = 9x^2 - 48x + 64$

$\Rightarrow \quad 9x^2 - 56x + 55 = 0$

$\Rightarrow \quad (9x-11)(x-5) = 0$

$\Rightarrow \quad 9x - 11 = 0$ or $x - 5 = 0$

$x = \frac{11}{9}$ $\qquad x = 5$ (given)

Substitute $x = \frac{11}{9}$ into (3): $y = 3\left(\frac{11}{9}\right) - 8$

$= -\frac{13}{3}$

$= -4\frac{1}{3}$

$\therefore$ $x = \frac{11}{9}$, $y = -4\frac{1}{3}$ **(Ans)**

3. Let the 2 positive numbers be x and y.

$x - y = 5$ .... (1)

$x^2 + y^2 = 377$ .... (2)

From (1): $x = 5 + y$ .... (3)

Substitute (3) into (2):

$(5+y)^2 + y^2 = 377$

$\Rightarrow \quad 25 + 10y + y^2 + y^2 = 377$

$\Rightarrow \quad 2y^2 + 10y - 352 = 0$

$\Rightarrow \quad y^2 + 5y - 176 = 0$

$\Rightarrow \quad (y+16)(y-11) = 0$

$\Rightarrow \quad y + 16 = 0$ or $y - 11 = 0$

$y = -16$ (rej.) $\qquad y = 11$

Substitute $y = 11$ into (3): $x = 5 + (11)$

$= 16$

$\therefore$ the 2 numbers are 16 and 11. **(Ans)**

4. $x + y = 3$ .... (1)

$x^2 - 2x + 2y^2 = 3$ .... (2)

From (1): $x = 3 - y$ .... (3)

Substitute (3) into (2):

$(3-y)^2 - 2(3-y) + 2y^2 = 3$

$\Rightarrow \quad 9 - 6y + y^2 - 6 + 2y + 2y^2 = 3$

$\Rightarrow \quad 3y^2 - 4y = 0$

$\Rightarrow \quad y(3y-4) = 0$

$\Rightarrow \quad y = 0$ or $3y - 4 = 0$

$y = \frac{4}{3}$

Substitute $y = 0$ into (3): $x = 3$

Substitute $y = \frac{4}{3}$ into (3): $x = 3 - \frac{4}{3}$

$= \frac{5}{3}$

$= 1\frac{2}{3}$

$\therefore$ the points of intersection are $(3, 0)$ and $\left(1\frac{2}{3}, 1\frac{1}{3}\right)$. **(Ans)**

5. $y = 2x - 3 \cdots\cdots\cdots (1)$

$3x^2 - y^2 - 3x - 5y + 6 = 0 \cdots\cdots\cdots (2)$

Substitute (1) into (2):

$$\begin{aligned} & 3x^2 - (2x-3)^2 - 3x - 5(2x-3) + 6 = 0 \\ \Rightarrow\ & 3x^2 - 4x^2 + 12x - 9 - 3x - 10x + 15 + 6 = 0 \\ \Rightarrow\ & x^2 + x - 12 = 0 \\ \Rightarrow\ & (x-3)(x+4) = 0 \end{aligned}$$

$\Rightarrow\ x - 3 = 0$ or $x + 4 = 0$

$x = 3 \qquad x = -4$

Substitute $x = 3$ into (1): $y = 2(3) - 3 = 3$

Substitute $x = -4$ into (1): $y = 2(-4) - 3 = -11$

$\therefore$ A is $(3, 3)$ and B is $(-4, -11)$

or A is $(-4, -11)$ and B is $(3, 3)$ **(Ans)**

6. Substitute $(2, 3)$ into the 2 equations:

$2a + 3b = 8 \cdots\cdots\cdots (1)$

$2a^2 - 3b^2 = -10 \cdots\cdots\cdots (2)$

From (1): $a = \dfrac{8-3b}{2} \cdots\cdots\cdots (3)$

Substitute (3) into (2):

$$\begin{aligned} & 2\left(\frac{8-3b}{2}\right)^2 - 3b^2 = -10 \\ \Rightarrow\ & \frac{64 - 48b + 9b^2}{2} - 3b^2 = -10 \\ \Rightarrow\ & 64 - 48b + 9b^2 - 6b^2 = -20 \\ \Rightarrow\ & 3b^2 - 48b + 84 = 0 \\ \Rightarrow\ & b^2 - 16b + 28 = 0 \\ \Rightarrow\ & (b-14)(b-2) = 0 \end{aligned}$$

$\Rightarrow\ b - 14 = 0$ or $b - 2 = 0$

$b = 14 \qquad b = 2$

Substitute $b = 14$ into (3): $a = \dfrac{8 - 3(14)}{2} = -17$

Substitute $b = 2$ into (3): $a = \dfrac{8 - 3(2)}{2} = 1$

$\therefore$ $a = 1,\ b = 2$ or $a = -17,\ b = 14$ **(Ans)**

TOPIC 6

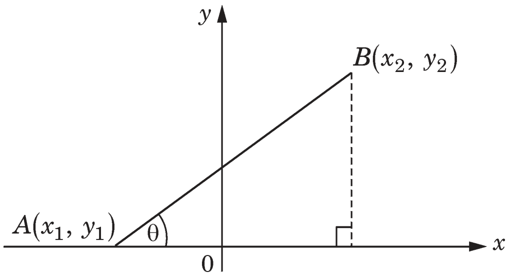
COORDINATE GEOMETRY

OVERVIEW OF THE TOPIC

COORDINATE GEOMETRY *6 key areas*	GRADIENT CONCEPT	
	ABOUT STRAIGHT LINES	• Mid-point • Distance formula • Collinearity • Perpendicular lines
	AREA OF POLYGONS	
	EQUATIONS OF STRAIGHT LINES	
	STRAIGHT LINE GRAPHS	• Linear Law • Non-linear functions
	FURTHER COORDINATE GEOMETRY	• Equation of a circle • Graphs of $y = ax^n$ and $y^2 = kx$

GRADIENT CONCEPT

Objectives:
- Define the gradient of a line segment via its end-points
- Understand the relations between the slope of a given line and the x and y intercepts



☑ For any 2 given points $A(x_1, y_1)$ and $B(x_2, y_2)$, the gradient of the line AB is:

Gradient of a line (2 given points)	$m_{AB} = \dfrac{y_2 - y_1}{x_2 - x_1}$ or $\dfrac{y_1 - y_2}{x_1 - x_2}$	y coordinates always on top of x coordinates

☑ θ is the angle of slope; the angle between the line and the positive direction of x-axis.

To find angle θ	$\tan\theta = m$	m is the gradient

☑ Remember these:

❶ Gradient of horizontal line = 0.

EXAMPLE

$$\underbrace{\text{Gradient of } (-1,\ 1) \text{ and } (2,\ 1)}_{\text{2 points with same } y \text{ value gives a horizontal line}} = \frac{1-1}{2-(-1)} = \frac{0}{3} = 0$$

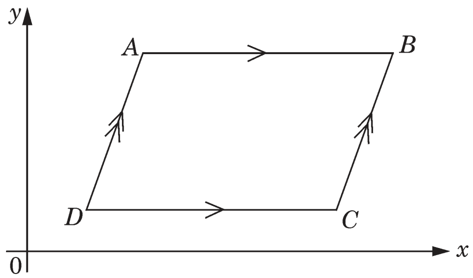
$\tan\theta = 0$

$\theta = 0°$ or $180°$ ← a horizontal line

❷ Gradient of vertical line is ∞.

❸ Parallel lines have same gradient.

EXAMPLE



AB and DC have same gradient.

AD and BC have same gradient.

STOP AND THINK

[Q] The line joining $(-1,-1)$ and $(p,0)$ is parallel to the line joining $(p,-2)$ and $(11,2)$. Find the value of p.

SOLUTION

Gradient of $(-1,-1)$ and $(p,0)$ = Gradient of $(p,-2)$ and $(11,2)$:

$$\Rightarrow \quad \frac{0-(-1)}{p-(-1)} = \frac{2-(-2)}{11-p}$$

$$\Rightarrow \quad \frac{1}{p+1} = \frac{4}{11-p}$$

$$\Rightarrow \quad 11-p = 4p+4$$

$$\Rightarrow \quad 5p = 7$$

$$\therefore \quad p = 1\frac{2}{5} \quad \textbf{(Ans)}$$

STOP AND THINK

[Q] Two lines are drawn from $P(-1,-3)$, one to $Q(4,2)$ and the other to $R(-4,2)$. What are their angles of slope?

SOLUTION

$$\text{Gradient of } PQ = \frac{2-(-3)}{4-(-1)} = \frac{5}{5} = 1$$

$$\tan\theta = 1$$
$$\theta = 45°$$

$\therefore$ PQ is sloped $45°$ to the positive x-axis. **(Ans)**

$$\text{Gradient of } QR = \frac{2-2}{-4-4} = 0$$

$\therefore$ QR is parallel to the x-axis. **(Ans)**

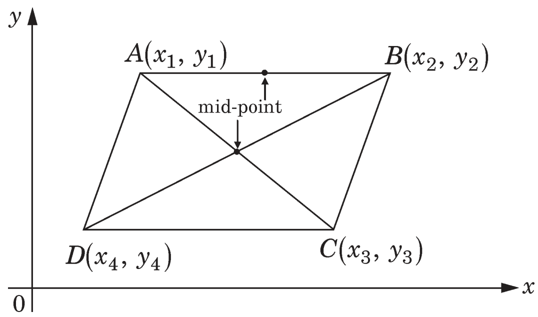
$$\text{Gradient of } PR = \frac{2-(-3)}{-4-(-1)} = \frac{5}{-3}$$

$$\tan\theta = \frac{5}{-3}$$
$$\theta = 121°$$

$\therefore$ PR is sloped $121°$ to the positive x-axis. **(Ans)**

MID-POINT

Objective: Find the mid-point of any 2 given points



☑ The mid-point of any line segment formed by 2 points $A(x_1, y_1)$ and $B(x_2, y_2)$ is:

Mid-point of a line (2 given points)	$\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$

☑ Given a parallelogram $ABCD$ with $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$, then mid-point of AC = mid-point of BD.

Mid-point of a parallelogram	mid-point of AC = mid-point of BD $\Rightarrow \left(\dfrac{x_1+x_3}{2}, \dfrac{y_1+y_3}{2}\right) = \left(\dfrac{x_2+x_4}{2}, \dfrac{y_2+y_4}{2}\right)$

DISTANCE FORMULA

Objective: Find the length of any line segment joined at its 2 given points

☑ Distance between 2 given points $A(x_1, y_1)$ and $B(x_2, y_2)$ is also known as the magnitude of AB or the length of AB.

Distance of a line (2 given points)	$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

COLLINEARITY

Objective: State the conditions of collinearity

☑ When 3 or more points lie on a single straight line, we say that the points are collinear. If 3 points A, B and C are collinear, then:

Collinearity (3 given points)	gradient of AB = gradient of BC = gradient of AC

STOP AND THINK

[Q] If the points $(-3,-2)$, $(-1, p-2)$ and $(p, 7)$ are collinear, find the two possible values of p.

SOLUTION

gradient of $(-1, p-2)$ and $(-3,-2)$ = gradient of $(p, 7)$ and $(-3,-2)$

$$\frac{p-2-(-2)}{-1-(-3)} = \frac{7-(-2)}{p-(-3)}$$

$$\Rightarrow \quad \frac{p}{2} = \frac{9}{p+3}$$

$$\Rightarrow \quad p(p+3) = 18$$

$$\Rightarrow \quad p^2 + 3p - 18 = 0$$

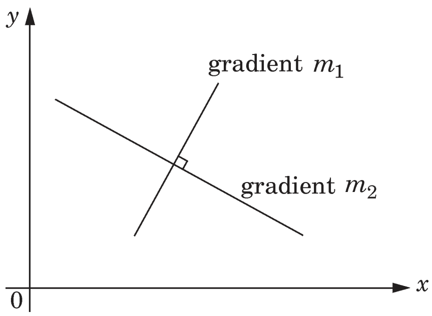
$$\Rightarrow \quad (p-3)(p+6) = 0$$

$p-3=0$ or $p+6=0$

$p=3$ **(Ans)** or $p=-6$ **(Ans)**

PERPENDICULAR LINES

Objectives:
- Identify the characteristics of perpendicular lines
- Make use of the characteristics of perpendicular lines to solve problems



☑ If m_1 and m_2 are gradients of two perpendicular lines (intersect at 90°), then:

Gradients of 2 perpendicular lines	$m_1 \times m_2 = -1$ or $m_1 = -\dfrac{1}{m_2}$	where $m_1 \neq 0$, $m_2 \neq 0$

EXAM TIP

A **parallelogram** has parallel opposite sides of the same length but the adjacent sides are NOT perpendicular to each other.

A **square** has 4 equal sides (parallel opposite sides) and are perpendicular to each other.

A **rhombus** also has 4 equal sides but with ONLY perpendicular lines crossing at its mid-point.

A **perpendicular bisector** cuts another line at its mid-point at 90°.

STOP AND THINK

[Q] Find the gradient of a line perpendicular to the line joining the points $(a, 3a)$ and $(2a, -a)$.

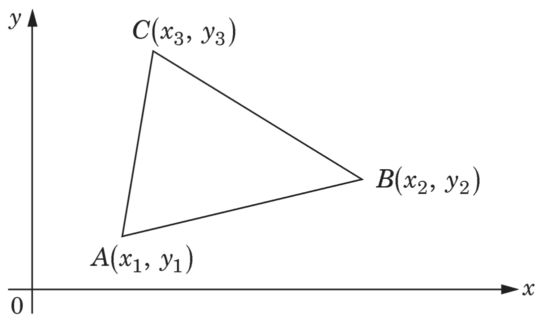
SOLUTION

$$\text{Gradient of } (a, 3a) \text{ and } (2a, -a) = \frac{3a-(-a)}{a-(2a)} = \frac{4a}{-a} = -4$$

$$\therefore \text{ Gradient of} \perp \text{line} = -\frac{1}{-4} = \frac{1}{4} \quad \textbf{(Ans)}$$

AREA OF POLYGONS

Objective: Apply the general formula for finding area of a polygon given the coordinates of the vertices



☑ Given vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, the area of ΔABC is

Area of ΔABC	$\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$ $= \frac{1}{2}\left|(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\right|$

arrange the 3 points in anti-clockwise direction ends with the start point

arrange $\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$ in cross pattern |\\\\\\|–|///|

EXAM TIP

This formula is applicable to all types of polygons.
Note that the vertices must be arranged in anti-clockwise direction.

STOP AND THINK

[Q] Three points have coordinates $O(0,0)$, $A(5,0)$ and $B(7,6)$. If P is the point (x, y), where $y>0$, calculate the value of x and of y given that $AP = BP$, and that the area of ΔAOP is 10 square units.

SOLUTION

$$AP = \sqrt{(5-x)^2+(0-y)^2}$$
$$= \sqrt{(5-x)^2+y^2}$$
$$BP = \sqrt{(7-x)^2+(6-y)^2}$$

$$AP = BP$$
$$\Rightarrow \quad \sqrt{(5-x)^2+y^2} = \sqrt{(7-x)^2+(6-y)^2}$$
$$\Rightarrow \quad (5-x)^2+y^2 = (7-x)^2+(6-y)^2$$
$$\Rightarrow \quad 25-10x+x^2+y^2 = 49-14x+x^2+36-12y+y^2$$
$$\Rightarrow \quad 25-10x = 85-14x-12y$$
$$\Rightarrow \quad 4x+12y = 60$$
$$\Rightarrow \quad x+3y = 15 \cdots\cdots\cdots (1)$$

$$\text{Area of } \Delta AOP = 10 \text{ units}^2$$
$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} 5 & 0 & x & 5 \\ 0 & 0 & y & 0 \end{vmatrix} = 10$$
$$\Rightarrow \quad \frac{1}{2}|0-5y| = 10$$
$$\Rightarrow \quad 5y = 20 \qquad (\text{since } y>0)$$
$$\therefore \quad y = 4 \quad \textbf{(Ans)}$$

Substitute $y=4$ into (1): $x+3(4)=15$

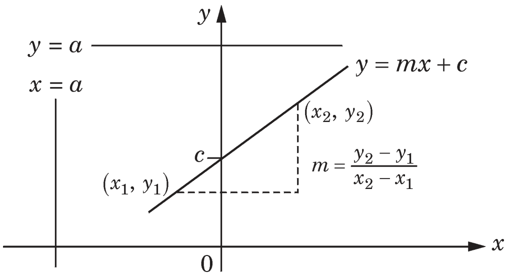
$$\therefore \quad x = 3 \quad \textbf{(Ans)}$$

EQUATIONS OF STRAIGHT LINES

Objective: Use necessary information to form linear equations

☑ The general form of a linear equation (a straight line) is:

Linear Equation	$y = mx + c$	where m is the gradient, c is y-intercept



By knowing any 2 points of a straight line, we can:

❶ find the gradient, m

❷ substitute any 1 point and value of m into $y = mx + c$ to find c

❸ find the equation

☑ Horizontal lines have equations of the form $y = a$. ($y = 0$ is the x-axis.)

☑ Vertical lines have equations of the form $x = a$. ($x = 0$ is the y-axis.)

EXAM TIP

There are 3 ways to present linear functions:

(1) General form: $ax + by + c = 0$

(2) Gradient-intercept form: $y = mx + c$

(3) Double intercept form: $\frac{x}{a} + \frac{y}{b} = 1$

STOP AND THINK

[Q] Find the equation of the line perpendicular to the line $2y - 3x + 6 = 0$ and passing through the point $(6, 1)$.

SOLUTION

$$2y - 3x + 6 = 0$$
$$\Rightarrow \quad 2y = 3x - 6$$
$$\Rightarrow \quad y = \frac{3}{2}x - 3$$

Gradient of line $= \frac{3}{2}$

Gradient of $\perp$ line $= -\frac{2}{3}$

Equation of $\perp$ line: $y = -\frac{2}{3}x + c$

Substitute $(6, 1)$ into $y = -\frac{2}{3}x + c$: $1 = -\frac{2}{3}(6) + c$

$1 = -4 + c$

$c = 5$

$\therefore$ Equation of $\perp$ line is $y = -\frac{2}{3}x + 5$ **(Ans)**

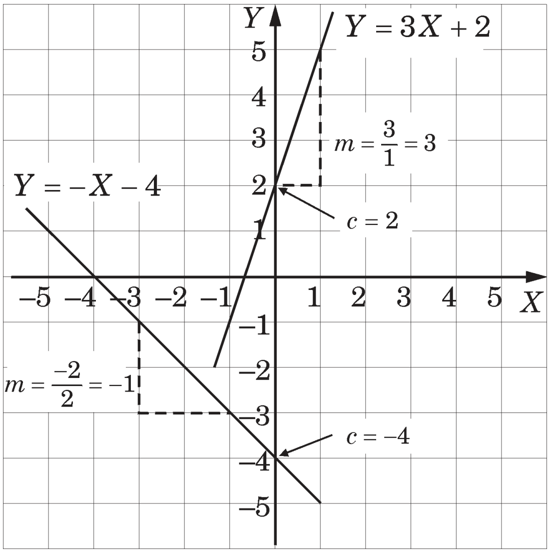
STRAIGHT LINE GRAPHS

Objectives:
- Convert straight line equations into equations of higher order
- Reduce non-linear functions to the linear form $y = mx + c$

Linear Law

☑ For two variables X and Y related by the linear equation $\boxed{Y = mX + c}$, the graph is always a straight line with m as the gradient and c the Y-intercept.

EXAMPLE



Non-linear functions

☑ Any functions <u>not</u> in the form of $y = mx + c$ are considered as non-linear functions. These functions are not straight lines, their variables x and y do not follow a linear relationship.

☑ Linear Law can be applied by expressing a non-linear equation in the form $Y = mX + c$, where X and Y are expressions in x and/or y, and m and c are expressions in a and b (the unknown constants) respectively.

☑ Now a straight line can be plotted and the <u>values</u> of the unknown constants a and b can be obtained from:

❶ the graph, or

❷ by substituting the values of two ordered pairs in the equation and solving the two equations in m and c simultaneously.

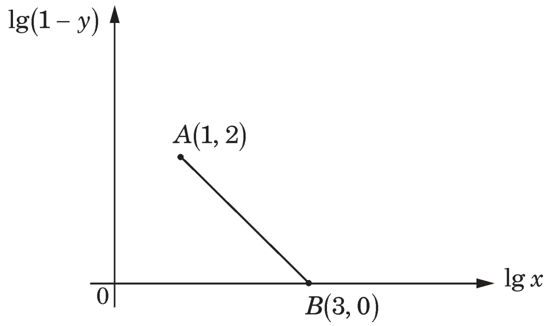
☑ The following table shows some examples of non-linear functions being converted to linear form: $Y = mX + c$:

	Non-linear functions $\xrightarrow{\text{(converted to)}} Y = mX + c$	X	Y	m	c
1)	$y = ax^n + b$	x^n	y	a	b
2)	$y = \frac{a}{x^n} + b$	$\frac{1}{x^n}$	y	a	b
3)	$\frac{1}{y} = ax^n + b$	x^n	$\frac{1}{y}$	a	b
4)	$y = ax^2 + bx \Rightarrow \boxed{\frac{y}{x} = ax + b}$	x	$\frac{y}{x}$	a	b
5)	$ax^2 + by^3 = x \Rightarrow \boxed{\frac{y^3}{x} = \frac{1}{b} - \frac{a}{b}x}$	x	$\frac{y^3}{x}$	$-\frac{a}{b}$	$\frac{1}{b}$
6)	$\frac{a}{x} + \frac{b}{y} = n \Rightarrow \boxed{\frac{1}{y} = \frac{n}{b} - \frac{a}{b}\left(\frac{1}{x}\right)}$	$\frac{1}{x}$	$\frac{1}{y}$	$-\frac{a}{b}$	$\frac{n}{b}$
7)	$y = \frac{a}{x-b} \Rightarrow \boxed{\frac{1}{y} = \frac{1}{a}x - \frac{b}{a}}$	x	$\frac{1}{y}$	$\frac{1}{a}$	$-\frac{b}{a}$
8)	$xy = \frac{a}{x} + bx \Rightarrow \boxed{x^2y = bx^2 + a}$	x^2	x^2y	b	a
9)	$y = ax^2 + bx + c \Rightarrow \boxed{\frac{y-c}{x} = ax + b}$ (c is a known constant)	x	$\frac{y-c}{x}$	a	b
10)	$y = ax^b \Rightarrow \boxed{\lg y = b\lg x + \lg a}$	$\lg x$	$\lg y$	b	$\lg a$
11)	$y = ab^x \Rightarrow \boxed{\lg y = x\lg b + \lg a}$	x	$\lg y$	$\lg b$	$\lg a$
12)	$y = n - ax^b \Rightarrow \boxed{\lg(n-y) = b\lg x + \lg a}$ (n is a known constant)	$\lg x$	$\lg(n-y)$	b	$\lg a$

Plot Y against X
to obtain a straight line

STOP AND THINK

[Q] The diagram shows part of a straight line $A(1,2)$ and $B(3,0)$ drawn to represent the equation $y=1-ax^b$, where a and b are constants. Find the values of a and b.



SOLUTION

$$\text{Gradient of the line} = \frac{2-0}{1-3} = -1$$

Equation of the line: $\lg(1-y) = -\lg x + c$

$$\Rightarrow \quad 2 = -1 + c$$
$$c = 3$$

$$\lg(1-y) = -\lg x + 3$$
$$\Rightarrow \quad \lg(1-y) + \lg x = 3$$
$$\Rightarrow \quad \lg(1-y)x = 3$$
$$\Rightarrow \quad (1-y)x = 10^3$$
$$\Rightarrow \quad (1-y)x = 1000$$
$$\Rightarrow \quad x - xy = 1000$$
$$xy = x - 1000$$
$$y = 1 - \frac{1000}{x}$$
$$y = 1 - 1000x^{-1}$$

since $y = 1 - ax^b$,

$a = 1000$ **(Ans)**

$b = -1$ **(Ans)**

STOP AND THINK

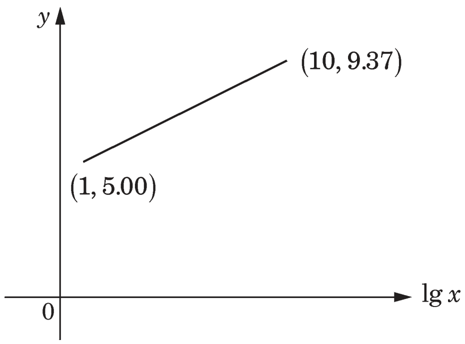
[Q] (a) The table shows experimental values of two variables u and v.

u	1	2	3	4	5
v	5.9	5.6	5.8	6.0	6.2

It is known that u and v are related by an equation of the form $u = \left(\frac{au+b}{v}\right)^2$, where a and b are constants.

Draw the graph of $v\sqrt{u}$ against u for the given data and use it to evaluate the value of a and of b.

(b) The diagram shows part of a straight line graph drawn to represent the equation $x^a = 10^{y-b}$. Calculate the value of a and of b.



SOLUTION

(a)

$$u = \left(\frac{au+b}{v}\right)^2$$

$$\sqrt{u} = \frac{au+b}{v}$$

$$v\sqrt{u} = au+b$$

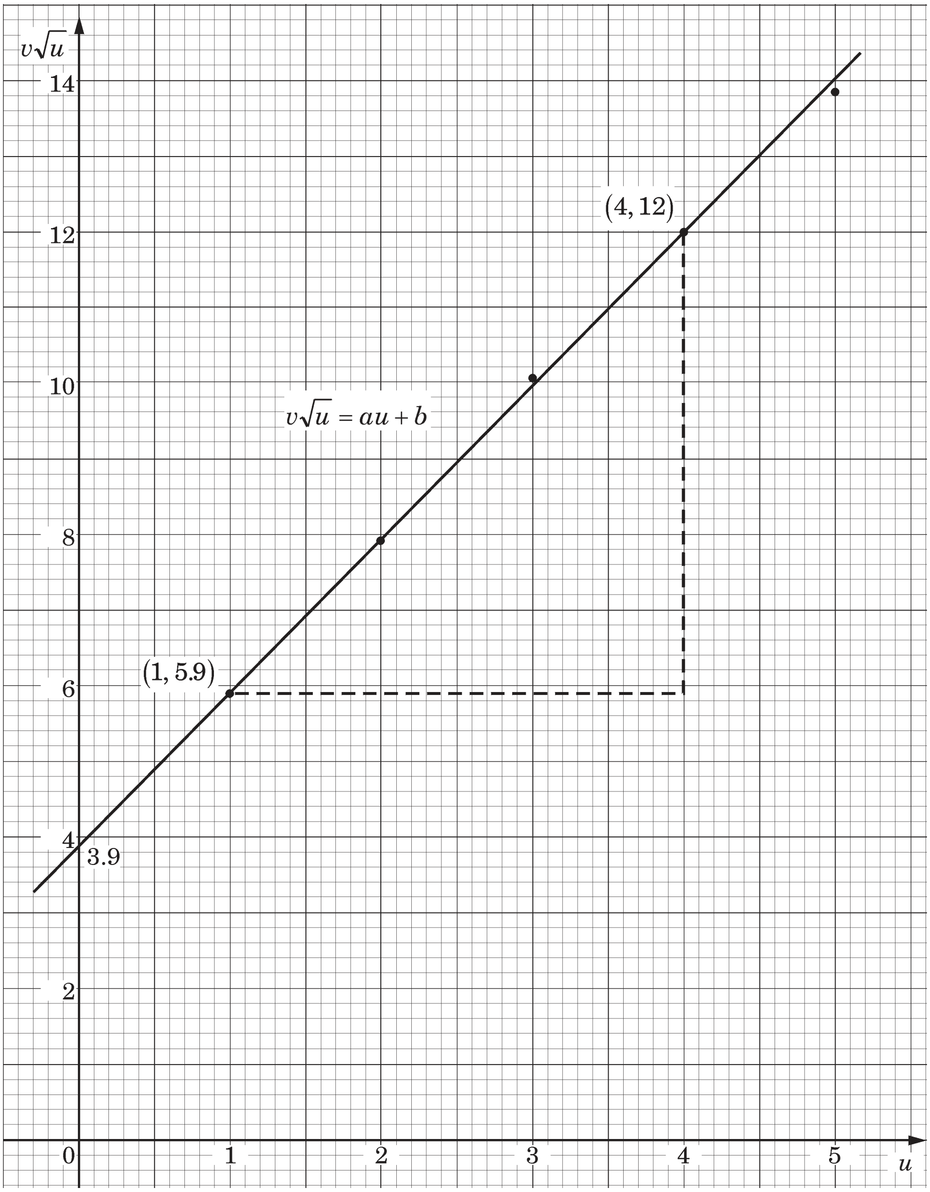
Plot $v\sqrt{u}$ against u.
Gradient = a
intercept = b

u	1	2	3	4	5
$v\sqrt{u}$	5.9	7.92	10.05	12	13.86

From the graph:

$$a = \frac{12-5.9}{4-1}$$
≈ 2.03 (3 sf) **(Ans)**

$b \approx 3.9$ **(Ans)**



(b) Gradient of line $= \dfrac{9.37-5}{10-1} = \dfrac{437}{900}$

Equation of the line:

$$y = \frac{437}{900}\lg x + c$$

$$5 = \frac{437}{900} + c \quad \Rightarrow \quad c = 4\frac{463}{900}$$

$$\therefore \; y = \frac{437}{900}\lg x + 4\frac{463}{900}$$

$$x^a = 10^{y-b}$$

$$\lg x^a = \lg 10^{y-b}$$

$$a\lg x = y - b$$

$$y = a\lg x + b$$

By comparison,

$a = \dfrac{437}{900}$ **(Ans)**

$b = 4\dfrac{463}{900}$ **(Ans)**

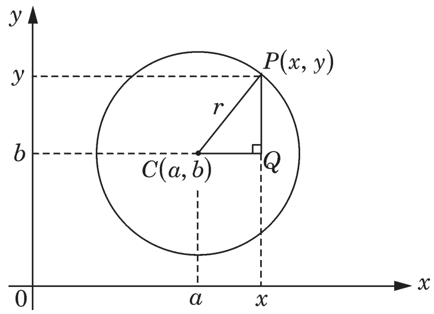
FURTHER COORDINATE GEOMETRY

Objectives:
- Find the equation of a circle given the centre and radius, and vice versa
- Draw the graphs of $y = ax^n$ and $y^2 = kx$

Equation of a circle

☑ The circle is defined as the locus of a point moving at a given distance (the radius) from a fixed point (the centre of the circle).

☑ Let $P(x, y)$ be any point along the circumference of a circle with centre (a, b) and radius r on the x-y plane.



From right-angled ΔCQP:

$CQ^2 + QP^2 = CP^2$

$\Rightarrow \ (x-a)^2 + (y-b)^2 = r^2$

This is the same as applying Distance Formula, but the purpose is not to find the distance of the radius r.
When (a, b) is a fixed point and with known radius, changing the values of (x, y) will give the circumference of a circle.

☑

<table>
<tr><td>Circle Formula</td><td>$(x-a)^2 + (y-b)^2 = r^2$</td><td>with <u>given</u> centre (a, b) and radius r, apply this formula to find equation of a circle</td></tr>
</table>

☑ If a circle has the origin $(0, 0)$ as the point of centre (a, b), then the equation is $\boxed{x^2 + y^2 = r^2}$

☑

<table>
<tr><td><u>General Equation of a circle</u></td><td>$\boxed{x^2 + y^2 + 2ax + 2by + d = 0}$ Notes:
(1) coefficient of x^2 and y^2 must be the same
(2) no 'xy' term</td></tr>
<tr><td>rearrange to find centre and radius</td><td>$\Rightarrow \ (x+a)^2 + (y+b)^2 - a^2 - b^2 + d = 0$
$\Rightarrow \ (x+a)^2 + (y+b)^2 = \underbrace{a^2 + b^2 - d}$
$\therefore$ centre is $(-a, -b)$, radius $= \sqrt{a^2 + b^2 - d}$</td></tr>
</table>

STOP AND THINK

[Q] (a) Find the equation of the circle with centre $(2,1)$ and radius $= 4$ units.

(b) Find the centre and radius of a circle whose equation is $x^2 + y^2 - 6x + 4y + 4 = 0$.

SOLUTION

(a) Substitute $(2,1)$ and $r = 4$ into circle formula:

$$(x-2)^2 + (y-1)^2 = (4)^2$$
$$\Rightarrow \quad x^2 - 4x + 4 + y^2 - 2y + 1 = 16$$
$$\Rightarrow \quad x^2 + y^2 - 4x - 2y - 11 = 0 \quad \textbf{(Ans)}$$

(b)
$$x^2 + y^2 - 6x + 4y + 4 = 0$$
$$\Rightarrow \quad x^2 - 6x + y^2 + 4y + 4 = 0 \qquad \leftarrow \boxed{\text{'complete the square' method}}$$
$$\Rightarrow \quad (x-3)^2 - (3)^2 + (y+2)^2 - (2)^2 + 4 = 0$$
$$\Rightarrow \quad (x-3)^2 + (y+2)^2 = (3)^2$$

$\therefore$ centre is $(3, -2)$ and radius $= 3$ units **(Ans)**

STOP AND THINK

[Q] A diameter of a circle $x^2 + y^2 - 4x - 2y + 3 = 0$ passes through $(1,3)$. Find the equation of this diameter.

SOLUTION

$$x^2 + y^2 - 4x - 2y + 3 = 0$$
$$\Rightarrow \quad x^2 - 4x + y^2 - 2y + 3 = 0$$
$$\Rightarrow \quad (x-2)^2 - 4 + (y-1)^2 - 1 + 3 = 0$$
$$\Rightarrow \quad (x-2)^2 + (y-1)^2 - 2 = 0$$
$$\Rightarrow \quad (x-2)^2 + (y-1)^2 = \left(\sqrt{2}\right)^2$$

$\therefore$ centre is $(2,1)$

Since the diameter passes through $(2,1)$ and $(1,3)$,

$$\text{gradient of diameter} = \frac{3-1}{1-2} = -2$$

At $(1,3)$ and $m = -2$: $y = mx + c$

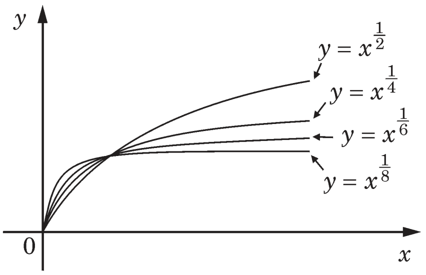
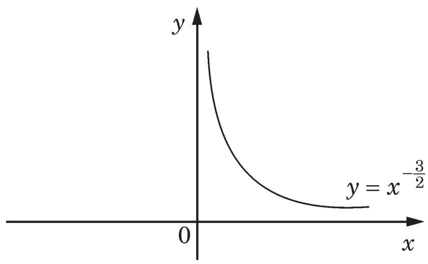
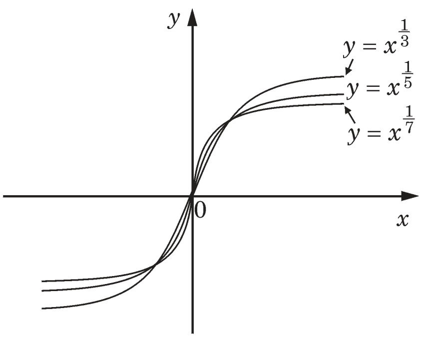
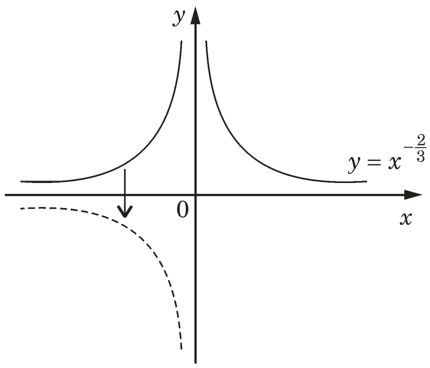
$3 = -2(1) + c$

$c = 5$

$\therefore$ Equation of diameter is $y = -2x + 5$ **(Ans)**

GRAPHS OF $y = ax^n$ AND $y^2 = kx$

☑ A power function $y = ax^n$ (where a is a constant >0 and n is a simple rational number such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$ ...etc) given as $\boxed{y = ax^{\frac{m}{n}}}$ has the following groups of graphs:

① $y = ax^{\frac{1}{n}}$ when n is even & positive, $m = 1$ $y = a\sqrt[n]{x} \Rightarrow x \geq 0$ (range of curve: $y \geq 0$)	② $y = ax^{\frac{m}{n}}$ when n is even & negative, m is odd $y = \dfrac{a}{\sqrt[n]{x^m}} \Rightarrow x \neq 0,\ x > 0$ (range of curve: $y > 0$)
	
③ $y = ax^{\frac{1}{n}}$ when n is odd & positive, $m = 1$ $y = a\sqrt[n]{x} \Rightarrow$ all x (range of curve: all y)	④ $y = ax^{\frac{m}{n}}$ when n is odd & negative, m is even $y = \dfrac{a}{\sqrt[n]{x^m}} \Rightarrow x \neq 0$, all x (range of curve: $y > 0$)
 (Note: The curves look like ① but with the negative reverse side.)	 (Note: If m=1, the curves will have the negative side in reverse position as shown.)

EXAM TIP

step ❶ let n decides domain of x first: $\begin{cases} \text{even} \Rightarrow \text{only } +\text{ve } x \\ \text{odd} \Rightarrow \text{all } x \end{cases}$

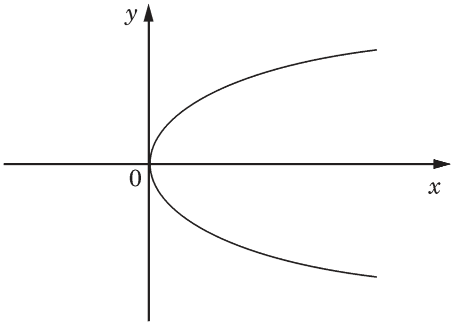
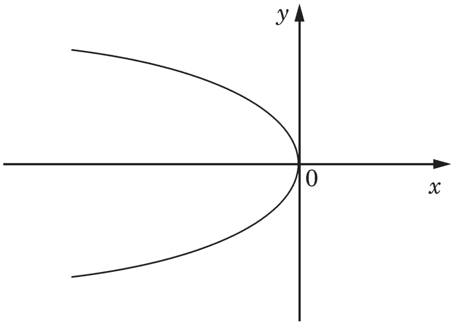
step ❷ sign of n decides pattern of curve: $\begin{cases} \text{positive} \Rightarrow \text{'}S\text{' pattern} \\ \text{negative} \Rightarrow \text{'}C\text{' pattern} \end{cases}$

step ❸ then m decides range of y: $\begin{cases} \text{even} \Rightarrow \text{only } +\text{ve } y \\ \text{odd} \Rightarrow \text{all } y \end{cases}$ note: should not overrule domain of x

☑ Parabola of basic equation $\boxed{y^2 = kx \;\; (k \neq 0)}$:

❶ The curve is symmetrical about the x-axis.

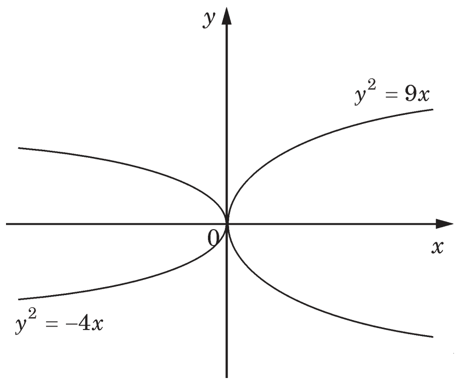
❷ The vertex is always at the origin $(0, 0)$.

① $y^2 = kx$ when $k > 0$, x must be positive	② $y^2 = kx$ when $k < 0$, x must be negative
	

STOP AND THINK

[Q] On the same xy-axis, sketch the graph of $y^2 = 9x$ and $y^2 = -4x$.

SOLUTION



(Ans)

TOPIC 14

TRIGONOMETRY

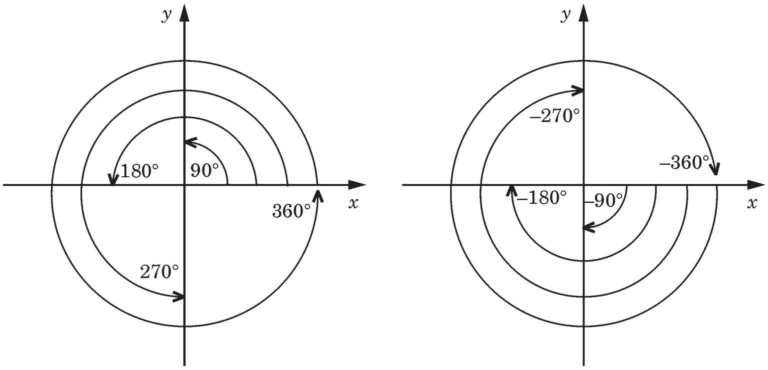
OVERVIEW OF THE TOPIC

TRIGONOMETRY *6 key areas*	GENERAL ANGLES	• General angles
		• Basic angles
	TRIGONOMETRIC RATIOS OF ANY ANGLES	• Definition of trigonometric ratios in 4 quadrants
		• Negative angles
		• Complementary angles
		• Finding values using special angles
	SOLVING TRIGONOMETRIC EQUATIONS	
	TRIGONOMETRIC GRAPHS	• Understanging basic trigo curves
		• Applications of trigo functions
	TRIGONOMETRIC IDENTITIES	• The reciprocal trigonometric functions
		• Relationships between sinθ, cosθ, tanθ
		• Pythagorean identities
		• Proving identities
	FURTHER TRIGONOMETRIC IDENTITIES	• Compound angle formulae
		• Double-angle formulae
		• Half-angle formulae
		• Factor formulae
		• R-formula

GENERAL ANGLES

Objective: Extend the knowledge of general and basic angles and relate them to trigonometric ratios

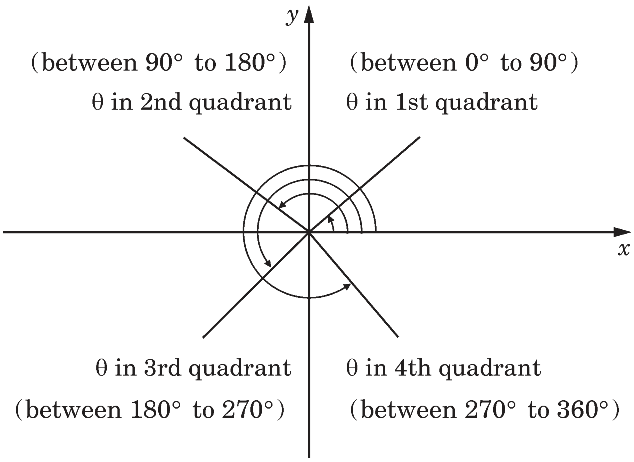
General angles



Let θ be the general angle from 0° to 360°

- ☑ In a cartesian plane, for an angle θ measured from the positive x-axis, θ is a positive angle if it is measured in an anti-clockwise direction. θ is a negative angle if it is measured in a clockwise direction.

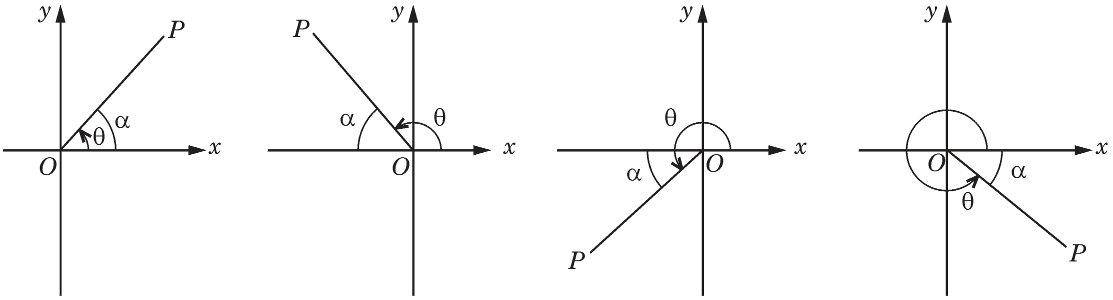
- ☑ These angles are known as general angles in the four quadrants.



Basic angles

☑ The angle, α, which is the positive acute angle (an angle between 0° and 90°) made by the arm OP and the x-axis, is known as the basic angle, associated angle or the reference angle.

☑ Every general angle (θ) has a corresponding basic angle (α).



θ in 1st quadrant, if we know α, then	θ in 2nd quadrant, if we know α, then	θ in 3rd quadrant, if we know α, then	θ in 4th quadrant, if we know α, then
$(\theta = \alpha)$	$(\theta = 180° - \alpha)$	$(\theta = 180° + \alpha)$	$(\theta = 360° - \alpha)$

STOP AND THINK

[Q] Find the basic angles, α, of the following positive general angles.

(a) $\theta = 160°$

(b) $\theta = 215°$

(c) $\theta = 312°$

SOLUTION

(a) 160° is in 2nd quadrant, basic angle $\alpha = 180° - 160° = 20°$ **(Ans)**

(b) 215° is in 3rd quadrant, basic angle $\alpha = 215° - 180° = 35°$ **(Ans)**

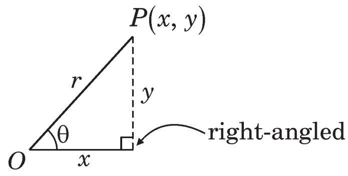
(c) 312° is in 4th quadrant, basic angle $\alpha = 360° - 312° = 48°$ **(Ans)**

TRIGONOMETRIC RATIOS OF ANY ANGLES

Objectives:
- Know the 3 trigonometric ratios (sine, cosine, tangent) and relate them to general and basic angles
- Evaluate the values of trigonometric ratios of special angles without using calculators

Definition of trigonometric ratios in 4 quadrants

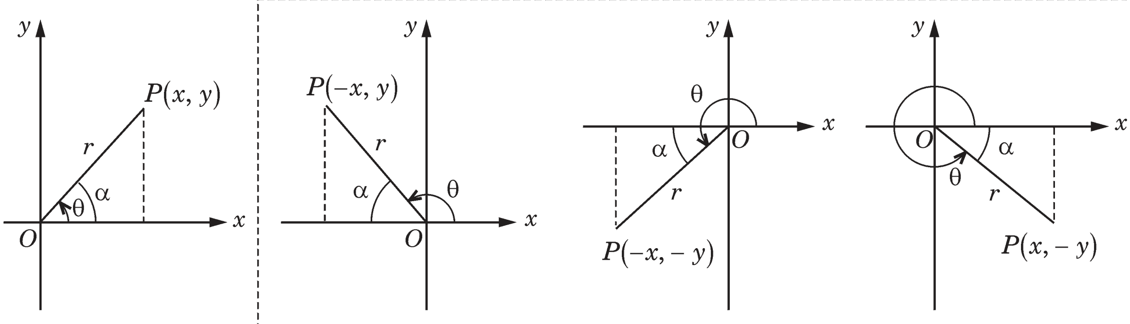
☑ We can define the 3 primary trigonometric ratios $\sin\theta$, $\cos\theta$ and $\tan\theta$ by using the right-angled triangle rule.



☑ Suppose a point $P(x, y)$ lies in the 1st quadrant where θ is acute and the length OP is r units. Then:

remember the basic rules:	$\sin\theta = \dfrac{y}{r}$ $\cos\theta = \dfrac{x}{r}$ $\tan\theta = \dfrac{y}{x}$	θ in 1st quadrant: all ratios have positive sign since x and y are positive

☑ However, if $P(x, y)$ lies in the other quadrants where θ is not acute (the signs of x and y depend on which quadrant they are in), then we need to use its corresponding basic angle, α, in order to define the 3 trigonometric ratios.

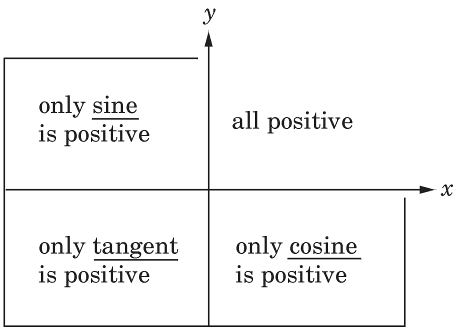


θ in 1st quadrant $\alpha = \theta$	θ in 2nd quadrant $\alpha = 180° - \theta$	θ in 3rd quadrant $\alpha = \theta - 180°$	θ in 4th quadrant $\alpha = 360° - \theta$
$\sin\theta = \dfrac{y}{r}$ $\cos\theta = \dfrac{x}{r}$ $\tan\theta = \dfrac{y}{x}$	$\sin\alpha = \dfrac{y}{r}$ $\cos\alpha = -\dfrac{x}{r}$ $\tan\alpha = -\dfrac{y}{x}$	$\sin\alpha = -\dfrac{y}{r}$ $\cos\alpha = -\dfrac{x}{r}$ $\tan\alpha = \dfrac{y}{x}$	$\sin\alpha = -\dfrac{y}{r}$ $\cos\alpha = \dfrac{x}{r}$ $\tan\alpha = -\dfrac{y}{x}$

☑ In general, with any angle θ and basic angle α, if θ is in:

①	1^{st} quadrant	$\sin\theta = \sin\alpha$ $\cos\theta = \cos\alpha$ $\tan\theta = \tan\alpha$	$\alpha=\theta$	$\theta=\alpha$
②	2^{nd} quadrant	$\sin\theta = \sin\alpha$ $\cos\theta = -\cos\alpha$ $\tan\theta = -\tan\alpha$	convert θ to α $\alpha=180°-\theta$	convert α to θ $\theta=180°-\alpha$
③	3^{rd} quadrant	$\sin\theta = -\sin\alpha$ $\cos\theta = -\cos\alpha$ $\tan\theta = \tan\alpha$	convert θ to α $\alpha = \theta - 180°$	convert α to θ $\theta=180°+\alpha$
④	4^{th} quadrant	$\sin\theta = -\sin\alpha$ $\cos\theta = \cos\alpha$ $\tan\theta = -\tan\alpha$	convert θ to α $\alpha = 360°-\theta$	convert α to θ $\theta=360°-\alpha$

☑ After reducing general angle θ to basic angle α, apply the following sign table:



STOP AND THINK

[Q] Find the 3 trigonometric ratios in terms of its basic angle of each of the following angles:

(a) $230°$

(b) $305°$

(c) $420°$

(d) $500°$

SOLUTION

(a) $\left(\theta \text{ in } 3^{\text{rd}} \text{ quadrant, only tangent is } +\text{ve}, \ \alpha = 230° - 180° = 50°\right)$

$\sin 230° = -\sin 50°$ **(Ans)**

$\cos 230° = -\cos 50°$ **(Ans)**

$\tan 230° = \tan 50°$ **(Ans)**

(b) $\left(\theta \text{ in } 4^{\text{th}} \text{ quadrant, only cosine is } +\text{ve}, \ \alpha = 360° - 305° = 55°\right)$

$\sin 305° = -\sin 55°$ **(Ans)**

$\cos 305° = \cos 55°$ **(Ans)**

$\tan 305° = -\tan 55°$ **(Ans)**

(c) $420° > 360°$ $(\theta \text{ has completed 1 round and restarted again})$

$420° - 360° = 60°$ $\left(\theta \text{ in } 1^{\text{st}} \text{ quadrant, all } +\text{ve, basic angle is the same}\right)$

$\sin 420° = \sin 60°$ **(Ans)**

$\cos 420° = \cos 60°$ **(Ans)**

$\tan 420° = \tan 60°$ **(Ans)**

(d) $500° > 360°$

$500° - 360° = 140°$ $\left(\theta \text{ in } 2^{\text{nd}} \text{ quadrant, only sine is } +\text{ve}, \ \alpha = 180° - 140° = 40°\right)$

$\sin 500° = \sin 40°$ **(Ans)**

$\cos 500° = -\cos 40°$ **(Ans)**

$\tan 500° = -\tan 40°$ **(Ans)**

Negative angles

☑ For any angle θ with <u>negative</u> value, i.e. measured in clockwise direction, just remember the following and then reduce to basic angles:

⑤	$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$ $\tan(-\theta) = -\tan\theta$	applying the easy rules first, then reduce to basic angles

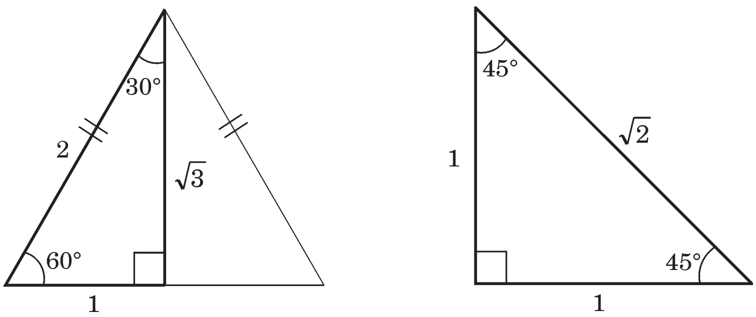
Complementary angles

☑

⑥	$\sin(90°-\theta) = \cos\theta$ $\cos(90°-\theta) = \sin\theta$ $\tan(90°-\theta) = \dfrac{1}{\tan\theta}$	easy rules for any acute angle θ (Refer to compound angle formulae)

Finding values using special angles (30°, 60°, 45°)

☑ The following angles have values that are easily remembered (without using calculator).



☑

(°) or (rad)	$\theta = 30°$ or $\frac{\pi}{6}$	$\theta = 60°$ or $\frac{\pi}{3}$	$\theta = 45°$ or $\frac{\pi}{4}$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	1

EXAM TIP

The table lists the values of all 3 trigonometric ratios for these special angles. You can also remember the ratios (of the length) of the 2 triangles and just apply the basic trigonometric rules.

STOP AND THINK

[Q] Find, without using a calculator, the values of the following:

(a) $\cos(-120°)$

(b) $\tan 480°$

(c) $\sin 750°$

SOLUTION

(a) $\cos(-120°) = \cos 120°$ $\left(2^{\text{nd}} \text{ quadrant, cosine is } -\text{ve}, \ \alpha = 60°\right)$

$= -\cos 60°$

$= -\dfrac{1}{2}$ **(Ans)**

(b) $480° > 360°$

$480° - 360° = 120°$ $\left(2^{\text{nd}} \text{ quadrant, tangent is } -\text{ve}, \ \alpha = 60°\right)$

$\tan 480° = -\tan 60°$

$= -\sqrt{3}$ **(Ans)**

(c) $750° > 720°$

$750° - 720° = 30°$

$\sin 750° = \sin 30°$

$= \dfrac{1}{2}$ **(Ans)**

STOP AND THINK

[Q] Find, without using a calculator, the exact value of

$$\cos 60° + \sin 120° - \cos 45° + \sin 225°.$$

SOLUTION

$$\begin{aligned}
&\cos 60° + \sin 120° - \cos 45° + \sin 225° \\
&= \cos 60° + \sin 60° - \cos 45° - \sin 45° \\
&= \frac{1}{2} + \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \\
&= \frac{1+\sqrt{3}}{2} - \frac{2}{\sqrt{2}} \\
&= \frac{1+\sqrt{3}}{2} - \frac{2\sqrt{2}}{2} \\
&= \frac{1+\sqrt{3}-2\sqrt{2}}{2} \quad \textbf{(Ans)}
\end{aligned}$$

SOLVING TRIGONOMETRIC EQUATIONS

Objective: Solve unknown angles (in degree or radians) when values of trigonometric ratios are given

☑ The procedure for obtaining the solution of the basic trigonometric equations of the form $\boxed{\sin\theta = k}$, etc are as follows:

❶ Identify all possible quadrants in which θ lies.
❷ Find the basic angle α.
❸ Find all the values of θ in the required interval.

EXAMPLE

To solve: $\sin\theta = \frac{1}{2}$

$\Rightarrow$ basic angle, $\alpha = \sin^{-1}\frac{1}{2}$

(from calculator) $= 30°$

Since $\sin\theta$ has positive value, another answer is from the 2nd quadrant: $180° - 30° = 150°$. $\therefore$ $\theta = 30°$ or $150°$

EXAM TIP

θ must be measured from the positive x-axis. Always round off the angles (in degrees) to 1 decimal place.

For angles in radian, the procedure of solving is the same. The only difference is to adjust the calculator to radian mode.

STOP AND THINK

[Q] Solve for $0° < x < 360°$, $\cos 2x = -0.34$.

SOLUTION

❶ Identify quadrants:

$\cos 2x$ is negative $\Rightarrow$ $2x$ lies in either 2nd or 3rd quadrant.

❷ Find basic angle:

From calculator, basic angle, $\alpha = \cos^{-1}(0.34)$
$= 70.12°$

❸ $0° < x < 360° \Rightarrow 0° < 2x < 720°$

$$\therefore\ 2x = 180° - 70.12°,\ 180° + 70.12°,\ 360° + (180° - 70.12°),\ 360° + (180° + 70.12°)$$
$$= 109.88°,\ 250.12°,\ 469.88°,\ 610.12°$$
$$x = 54.9°,\ 125.1°,\ 234.9°,\ 305.1°\ (1\text{ dp})$$ **(Ans)**

TRIGONOMETRIC GRAPHS

Objective: Draw the 3 basic trigo curves and extend the knowledge to transforming the trigo curves

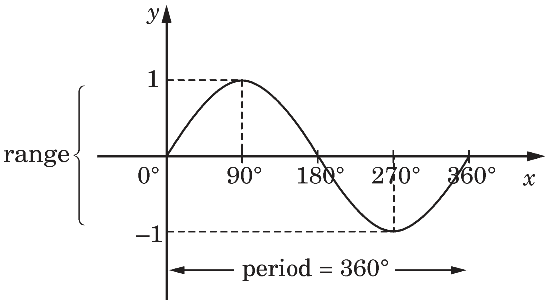
Understanding basic trigonometric curves:

☑ Graph of sine curve, ($y = \sin\theta$) :

Note: $-1 \le \sin\theta \le 1$

$\sin\theta$ has
max. value = 1
min. value = –1

When $\sin\theta = 0$
$\theta = 0°, 180°, 360°$

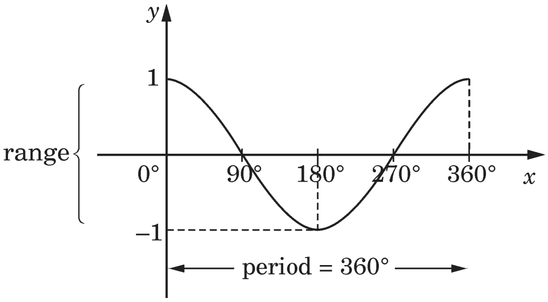


☑ Graph of cosine curve, ($y = \cos\theta$) :

Note: $-1 \le \cos\theta \le 1$

$\cos\theta$ has
max. value = 1
min. value = –1

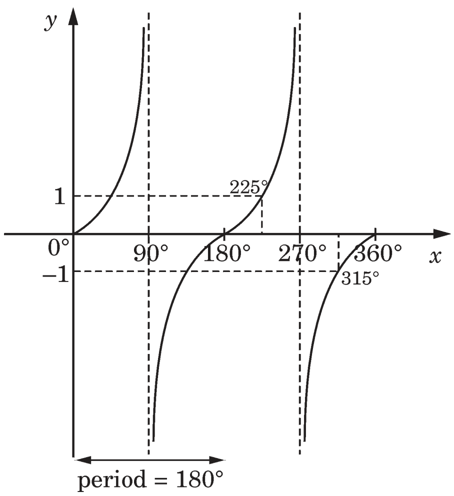
When $\cos\theta = 0$
$\theta = 90°, 270°$



☑ Graph of tangent curve, ($y = \tan\theta$) :

Notes:
- $\tan\theta$ can take any real value.
- $\tan\theta$ is undefined for $\theta = 90°, 270°$
- The curve approaches the lines $\theta = 90°$ and $\theta = 270°$. These lines are called asymptotes.

When $\tan\theta = 0$
$\theta = 0°, 180°, 360°$

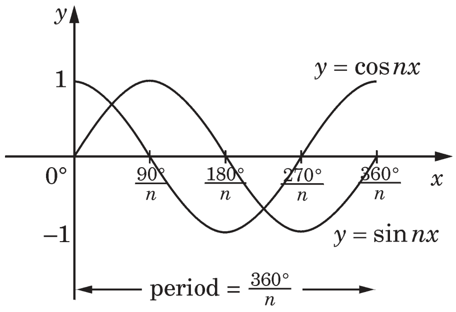


EXAM TIP

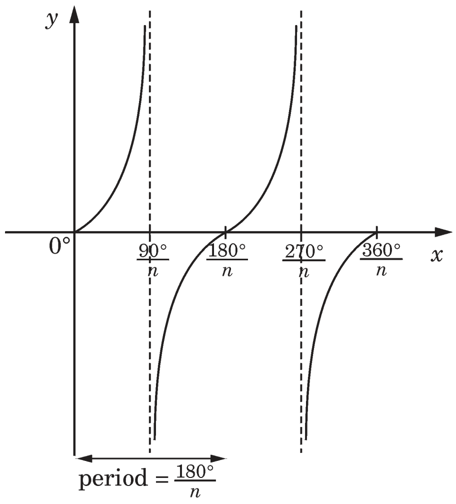
Just use calculator to plug in all the degrees and you can draw these 3 curves. The negative degrees have the same pattern.

Applications of trigonometric functions

☑ Graph of $y = \sin nx$, $y = \cos nx$, where $n > 0$:



☑ Graph of $y = \tan nx$, where $n > 0$:



EXAM TIP

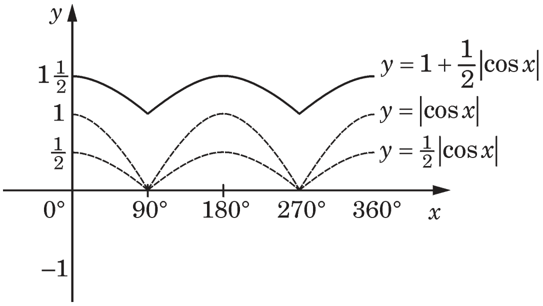
The basic trigonometric functions take the form of $y = \sin x$, $y = \cos x$ and $y = \tan x$. When changing x, only the periods are changed, the patterns remain the same (as shown above).

Besides changing x, the trigonometric functions can be in the form of $y = a\sin x + b$ (same for cosine and tangent) where a and b are real numbers. Here, the curves will transform and/or shift up or down along the y-axis.

STOP AND THINK

[Q] Sketch the graph of $y = 1 + \frac{1}{2}|\cos x|$ for the domain $0° \le x \le 360°$ and state the range corresponding to the given domain.

SOLUTION



Range of $y = 1 + \frac{1}{2}|\cos x|$ is $1 \le y \le 1\frac{1}{2}$ **(Ans)**