MATHEMATICS

KEY.POINTS

Exam Guide

O LEVEL

[16 Topics]

- Overview Of The Topic
- Objectives
- Stop & Think
- Challenging Questions
- Work Solutions

For O Levels

NEW SOLUTIONS. NEW MINDS.

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TOPIC 2

ARITHMETICAL PROBLEMS

OVERVIEW OF THE TOPIC

ARITHMETICAL PROBLEMS *4 key areas*	RATIO, RATE AND PROPORTION	• Ratio
		• Rate
		• Direct and inverse proportion
		• Maps and scales
	PERCENTAGES	
	SPEED	
	APPLICATIONS OF MATHEMATICS IN PRACTICAL SITUATIONS	• Simple interest and compound interest
		• Hire-Purchase
		• Discount, profit and loss
		• Money exchange

RATIO, RATE AND PROPORTION

Objectives: Demonstrate an understanding of (i) the elementary ideas and notion of ratio and divide a quantity in a given ratio, (ii) common measures of rate, (iii) direct and indirect proportion.

Ratio

- ☑ A ***ratio*** is used to compare two or more quantities of the same kind (or same unit).
- ☑ It shows how many of one quantity is compared to another.
- ☑ A ratio can be written with the ratio sign (:) or as a fraction.
- ☑ In general, the ratio of x to y, where x and y represent whole numbers is written as $\boxed{x : y \text{ or } \dfrac{x}{y}}$.
- ☑ The order in which the ratio is expressed is important. The quantity mentioned first must be written first, $\therefore\ x : y \neq y : x$.

EXAMPLE

❶ COMPARING TWO QUANTITIES OF THE SAME KIND

If the ratio of boys to girls of a school is 4 boys to 3 girls, then the number of boys and girls can be increased as follows:

$$\left.\begin{array}{c} \text{boys} : \text{girls} \\ 4 : 3 \\ 8 : 6 \\ 12 : 9 \\ 16 : 12 \\ \vdots \quad \vdots \quad \vdots \\ 480 : 360 \\ \vdots \quad \vdots \quad \vdots \end{array}\right\}$$

Formed by multiplying (or dividing) both side by a common factor, these are known as equivalent ratios.

4 : 3 is the ratio in its simplest form.

As a whole: 4 units + 3 units = 7 units

Now, a total of 840 students are to be divided based on the 4 : 3 ratio:

$$\begin{aligned} 4+3 &= 7 \text{ units} \\ 7 \text{ units} &\to 840 \\ \therefore\ 1 \text{ unit} &\to 840 \div 7 = 120 \\ \text{Boys get 4 units} &\to 4 \times 120 = 480 \quad \textbf{(Ans)} \\ \text{Girls get 3 units} &\to 3 \times 120 = 360 \quad \textbf{(Ans)} \end{aligned}$$

If the ratio of students to computers is 5 : 2, find the number of computers to be shared by 840 students:

students : computers

5 : 2

$\boxed{\begin{array}{l} 5 \text{ units} \to 840 \\ 1 \text{ unit} \to 840 \div 5 \\ = 168 \end{array}}$ $\quad \vdots \quad \vdots \quad \vdots \quad$ $\boxed{\begin{array}{l} 2 \text{ units} \times 168 \\ = 336 \quad \textbf{(Ans)} \end{array}}$

840 : number of computers (n)?

$$\frac{5}{2} = \frac{840}{n} \quad \leftarrow \boxed{\text{in fractional form}}$$

$$\therefore\ n = \frac{840 \times 2}{5}$$

$$= 336 \quad \textbf{(Ans)}$$

❷ COMPARING TWO QUANTITIES OF THE SAME UNIT

Find the ratio of 20 m to 500 cm in its simplest form.

20 m is equivalent to 2000 cm ← convert to same unit first

2000 : 500 = 20 : 5 = 4 : 1 **(Ans)**

❸ TO INCREASE (OR DECREASE) A QUANTITY IN THE RATIO $a : b$

a new quantity : b old quantity
120 : 100
6 : 5

To increase:
if we increase 100 teachers to 120 teachers, we say the number of teachers is increased in the ratio $\frac{120}{100} = \frac{6}{5}$

a new quantity : b old quantity
80 : 100
4 : 5

To decrease:
if we decrease 100 teachers to 80 teachers, we say the number of teachers is decreased in the ratio $\frac{80}{100} = \frac{4}{5}$

STOP AND THINK

[Q] An alloy, A, contains copper and zinc in the ratio 2 : 3.
An alloy, B, contains copper and tin in the ratio 13 : 7.
Find (a) the ratio of copper, zinc and tin in a new alloy containing equal weights of A and B.
(b) the weight of copper and zinc in the new alloy given that the new alloy weighs 440 kg.

SOLUTION

(a)

Alloy A	Alloy B
copper : zinc	copper : tin
2 : 3	13 : 7
$\frac{2}{5} : \frac{3}{5}$	$\frac{13}{20} : \frac{7}{20}$
$\frac{8}{20} : \frac{12}{20}$ ← same weight as alloy B	

The ratio of new alloy: copper : zinc : tin
8 + 13 : 12 : 7
21 : 12 : 7 **(Ans)**

(b) As a whole: 21 + 12 + 7 = 40

40 units → 440 kg

1 unit → $\frac{440}{40} = 11$ kg

∴ The weight of copper and zinc: (21+12) units → 33 × 11 kg
= 363 kg **(Ans)**

Rate

☑ A ***rate*** is a special ratio that compares two quantities which have DIFFERENT units.

EXAMPLE

A machine prints 100 pages in 4 minutes. Find its rate of printing for 1 minute and for 1 page.

For 1 minute, rate of printing $= \dfrac{\text{100 pages}}{\text{4 minutes}}$
$= 25$ pages/minute **(Ans)**

For 1 page, rate of printing $= \dfrac{\text{4 minutes}}{\text{100 pages}}$
$= 0.04$ minutes/page **(Ans)**

STOP AND THINK

[Q] Peter can fold paper planes at a rate of 3 planes per minute.
(a) How long does it take Peter to fold 36 planes?
(b) How many planes can Peter fold if he works for 20 seconds?

SOLUTION

(a) 3 planes $\rightarrow$ 1 min
1 plane $\rightarrow$ $\frac{1}{3}$ min
36 planes $\rightarrow$ $36 \times \frac{1}{3}$ min = 12 mins **(Ans)**

(b) 60 sec $\rightarrow$ 3 planes
1 sec $\rightarrow$ $\frac{1}{20}$ planes
20 sec $\rightarrow$ $20 \times \frac{1}{20}$ = 1 plane **(Ans)**

Direct and inverse proportion

☑ A ***proportion*** is an equation which shows that two ratios are equal.

☑ When two quantities are proportional, a change in one corresponds to a change in the other.

☑ In <u>direct</u> proportion, both quantities increase by the same factor, or both quantities decrease by the same factor.

☑

Direct Proportion	If y varies directly as x: $y \propto x$ then $y = kx$ (where k is a constant)

☑ In <u>inverse</u> proportion, when one quantity increases by a certain factor, the other quantity decreases by the same factor and vice versa.

☑

Inverse Proportion	If y varies indirectly as x: $$y \propto \frac{1}{x}$$ then $y = \frac{k}{x}$ (where k is a constant)

STOP AND THINK

[Q] If y varies inversely as $(2x + 1)$ and $y = 5$ when $x = 3$, find

(a) y when $x = 17$

(b) x when $y = 7$

SOLUTION

$$y \propto \frac{1}{2x+1}$$

$$\Rightarrow \quad y = \frac{k}{2x+1}$$

Substitute $y = 5$, $x = 3$ to solve k:

$$\Rightarrow \quad 5 = \frac{k}{6+1}$$

$$\therefore \quad k = 35$$

$$\therefore \quad y = \frac{35}{2x+1}$$

(a) When $x = 17$, $y = \frac{35}{34+1}$

$= 1$ **(Ans)**

(b) When $y = 7$,

$$7 = \frac{35}{2x+1}$$

$$\Rightarrow \quad 2x + 1 = 5$$

$$\Rightarrow \quad 2x = 4$$

$$\therefore \quad x = 2$$ **(Ans)**

STOP AND THINK

[Q] Given that y varies directly as $(x+2)(x+7)$ and $y=4$ when $x=1$, find the value of y when $x=5$.

SOLUTION

$$y \propto (x+2)(x+7)$$
$$\Rightarrow \quad y = k(x+2)(x+7)$$

Substitute $x=1,\ y=4$ to solve k:

$$\Rightarrow \quad 4 = k(1+2)(1+7)$$
$$\Rightarrow \quad 4 = k(3)(8)$$
$$\therefore \quad k = \frac{4}{24} = \frac{1}{6}$$

$$\therefore \quad y = \frac{1}{6}(x+2)(x+7)$$

When $x=5$, $y = \frac{1}{6}(7)(12)$
$= 14$ **(Ans)**

STOP AND THINK

[Q] The diameter of a sphere d is directly proportional to the cube root of its weight w. Given that the weight is 27 kg when the diameter is 21 cm, find the radius when the weight is 512 kg.

SOLUTION

$$d \propto \sqrt[3]{w}$$
$$\Rightarrow \quad d = k\sqrt[3]{w}$$

$w=27,\ d=21$, $\Rightarrow \quad 21 = k\sqrt[3]{27}$
$\Rightarrow \quad 21 = 3k$
$\therefore \quad k = 7$

$$\therefore \quad d = 7\sqrt[3]{w}$$

When $w=512$, $\Rightarrow \quad d = 7\times\sqrt[3]{512}$
$\Rightarrow \quad d = 7\times 8$
$= 56$

$\therefore$ radius $= 56 \text{ cm} \div 2$
$= 28 \text{ cm}$ **(Ans)**

Maps and scales

- A ***scale*** is a ratio that compares the size of a drawing to the actual size of the object.
- The linear scale of a map is usually expressed in the form $\boxed{1:n}$, where n is a whole number.
- The scale $1:n$ can also be expressed as a representative fraction (R.F.) of $\frac{1}{n}$.

EXAMPLE

A scale of 1 : 25 000 (linear) or $\frac{1}{25\,000}$ (R.F.) means the length of 1 cm on the drawing represents the distance of 25 000 cm or 250 m (100 cm = 1 m) or 0.25 km (1000 m = 1 km) in actual size.

- The area scale of a map is the square of its linear scale.
-

Area Scale	If the linear scale is $1:n$, then the area scale is $1:n^2$.

STOP AND THINK

[Q] (a) A map is drawn to a scale of 2 cm to 5 km. Find the actual distance between two towns where their distance apart on the map is

(i) 4.8 cm,

(ii) 36 cm.

(b) Two schools are 36 km apart. Find their distances apart on a map drawn to a scale of

(i) 2.5 cm to 10 km

(ii) $\frac{1}{2}$ cm to $2\frac{1}{4}$ km

(c) A map is drawn to a scale of 1 : 40 000.
Find the actual area, in square kilometres, of a factory warehouse whose area on the map is

(i) 180 cm^2

(ii) 20.5 cm^2

SOLUTION

(a) (i)

$$\begin{aligned}2\text{ cm} &\rightarrow 5\text{ km}\\1\text{ cm} &\rightarrow 5 \div 2\text{ km}\\&= 2.5\text{ km}\\4.8\text{ cm} &\rightarrow 4.8 \times 2.5\text{ km}\\&= 12\text{ km} \quad \textbf{(Ans)}\end{aligned}$$

(ii)

$$\begin{aligned}36\text{ cm} &\rightarrow 36 \times 2.5\text{ km}\\&= 90\text{ km} \quad \textbf{(Ans)}\end{aligned}$$

(b) (i)

$$10 \text{ km} \to 2.5 \text{ cm}$$
$$1 \text{ km} \to \frac{2.5}{10} \text{ cm}$$
$$\Rightarrow 36 \text{ km} \to 36 \times \frac{2.5}{10} \text{ cm}$$
$$= 9 \text{ cm} \quad \textbf{(Ans)}$$

(ii)

$$2\frac{1}{4} \text{ km} \to \frac{1}{2} \text{ cm}$$
$$1 \text{ km} \to \frac{1}{2} \div \frac{9}{4} \text{ cm}$$
$$= \frac{1}{2} \times \frac{4}{9} \text{ cm}$$
$$= \frac{2}{9} \text{ cm}$$
$$36 \text{ km} \to 36 \times \frac{2}{9} \text{ cm}$$
$$= 8 \text{ cm} \quad \textbf{(Ans)}$$

(c)

$$1 \text{ cm} \to 40\ 000 \text{ cm}$$
$$\Rightarrow 1 \text{ cm} \to 0.4 \text{ km}$$
$$\Rightarrow 1 \text{ cm} \to \frac{2}{5} \text{ km}$$
$$\Rightarrow 1 \text{ cm}^2 \to \frac{4}{25} \text{ km}^2$$

(i)

$$180 \text{ cm}^2 \to 180 \times \frac{4}{25} \text{ km}^2$$
$$= 28\frac{4}{5} \text{ km}^2$$
$$= 28.8 \text{ km}^2 \quad \textbf{(Ans)}$$

(ii)

$$20.5 \text{ cm}^2 \to 20.5 \times \frac{4}{25} \text{ km}^2$$
$$= 3.28 \text{ km}^2 \quad \textbf{(Ans)}$$

STOP AND THINK

[Q] The R.F. of a map is $\frac{1}{50\ 000}$.

(a) Calculate the actual distance, in kilometres, represented by 1 cm on the map.
(b) Find the actual length, in kilometres represented by 9.5 cm on the map.
(c) Find the area on the map, in square centimetres that represents an area of 60 km^2 on actual ground.

SOLUTION

(a) $1 \text{ cm} \to 50\ 000 \text{ cm}$

$\therefore$ 1 cm represents 0.5 km. **(Ans)**

(b)

$$1 \text{ cm} \to 0.5 \text{ km}$$
$$9.5 \text{ cm} \to 9.5 \times 0.5 \text{ km}$$
$$= 4.75 \text{ km} \quad \textbf{(Ans)}$$

(c)

$$1 \text{ cm} \to \frac{1}{2} \text{ km}$$
$$\Rightarrow 1 \text{ cm}^2 \to \frac{1}{4} \text{ km}^2$$
$$\Rightarrow 1 \text{ km}^2 \to 4 \text{ cm}^2$$
$$\Rightarrow 60 \text{ km}^2 \to 60 \times 4 \text{ cm}^2$$
$$= 240 \text{ cm}^2 \quad \textbf{(Ans)}$$

PERCENTAGES

Objectives: (i) Convert a percentage into a fraction or decimal and vice versa, (ii) express one quantity as a percentage of another and (iii) find percentage increase or decrease.

☑ A ***percentage*** is a special type of fraction in which the value given is a measure of the number of parts in every 100 parts that is to be used.

☑ The symbol % is used to represent percentage.

EXAMPLE

$$\underbrace{25\%}_{\text{percentage}} = \underbrace{\frac{25}{100} = \frac{1}{4}}_{\text{fraction}} = \underbrace{0.25}_{\text{decimal}} = \underbrace{25\%}_{\text{same as moving the decimal point 2 places to the right and add \%}}$$

☑

Applications of Percentage	Examples
① Converting a percentage into a fraction or decimal: ❶ divide by 100 to drop the % sign ❷ reduce the fraction to lowest term ❸ convert the fraction to a decimal (or move decimal point 2 places to the left and drop % sign)	$0.25\% = \frac{0.25}{100}$ $(= 0.0025)$ $= \frac{25}{10000}$ $= \frac{1}{400}$ $= 0.0025$
② Converting a fraction or a decimal into a percentage: ❶ multiply by 100 ❷ add %	$\frac{3}{2} = \frac{3}{2} \times 100\%$ $= 150\%$ $0.15 = 0.15 \times 100\%$ $= 15\%$
③ Find the percentage of a number: ❶ write the % as a fraction (i.e. divide by 100) ❷ multiply the number by the fraction	4.5% of 400 $= \frac{4.5}{100} \times 400$ $= 18$
④ Express one quantity as a percentage of another: ❶ write one quantity as a fraction of the other, making sure they are expressed in the same unit ❷ multiply the fraction by 100%	express 30 cm as a % of 5 m: $\frac{30}{500} \times 100\% = 6\%$
⑤ Find percentage increase or decrease: ❶ percentage increase $= \frac{\text{increase}}{\text{original amount}} \times 100\%$ ❷ percentage decrease $= \frac{\text{decrease}}{\text{original amount}} \times 100\%$	original amt = 80 new amt = 120 % increase $= \frac{40}{80} \times 100\%$ $= 50\%$

STOP AND THINK

[Q] (a) Express each of the percentage as a decimal.

(i) 0.68% (iii) 240%

(ii) $5\frac{4}{5}\%$

(b) Express each of the fraction as a percentage.

(i) $\frac{9}{25}$ (iii) $\frac{3}{8}$

(ii) $\frac{16}{40}$

(c) Express the first quantity as a percentage of the second.

(i) $1\frac{3}{4}$ years, 28 months

(ii) 80 cents, \$9.60

(iii) 150 cm^2, 18 m^2

(d) Evaluate each of the following:

(i) $22\frac{1}{2}\%$ of \$88 000

(ii) 150% of \$7240

(iii) 8% of 9 m 25 cm

SOLUTION

(a) (i) $0.68\% = \frac{0.68}{100} = 0.0068$ **(Ans)**

(ii) $5\frac{4}{5}\% = \frac{5.8}{100} = 0.058$ **(Ans)**

(iii) $240\% = \frac{240}{100} = 2.4$ **(Ans)**

(b) (i) $\frac{9}{25} = \frac{9}{25} \times 100\%$
$= 36\%$ **(Ans)**

(ii) $\frac{16}{40} = \frac{16}{40} \times 100\%$
$= 40\%$ **(Ans)**

(iii) $\frac{3}{8} = \frac{3}{8} \times 100\%$
$= 37.5\%$ **(Ans)**

(c) (i) $1\frac{3}{4}$ years $= \frac{7}{4} \times 12$ months
$= 21$ months

$\frac{21}{28} \times 100\% = 75\%$ **(Ans)**

TOPIC 12

MENSURATION

OVERVIEW OF THE TOPIC

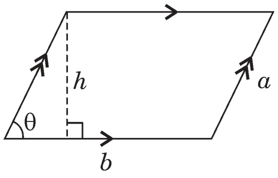
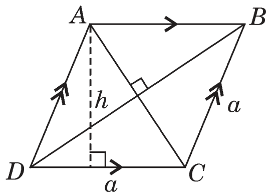
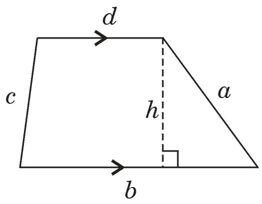
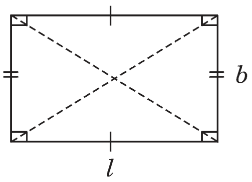
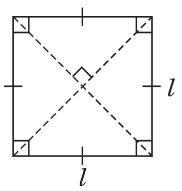
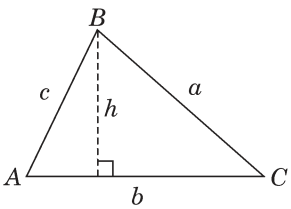
MENSURATION *2 key areas*	PERIMETER AND AREA	• Triangles and quadrilaterals
		• Circles
	SURFACE AREA AND VOLUME OF SOLIDS	• Cube and cuboid
		• Prism
		• Cylinder
		• Cone
		• Pyramid
		• Sphere

PERIMETER AND AREA

Objectives: Find the perimeter and area of triangles, quadrilaterals and circles. Solve problems involving the perimeter and area of triangles, quadrilaterals and circles.

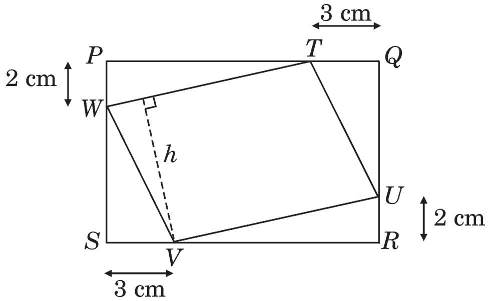
Triangles and quadrilaterals

- ☑ The ***perimeter*** of any closed figure is the total length of all the sides.
- ☑ To find ***area*** of quadrilaterals and triangles, apply formulae:

	**Parallelogram** ① Perimeter $= a+b+a+b = 2(a+b)$ ② Area $=$ base $\times$ height $= b \times h$ (or Area $= ab\sin\theta$)
	**Rhombus** ① Perimeter $= a+a+a+a = 4a$ ② Area $=$ base $\times$ height $= a \times h$ $\left(\text{or Area} = \frac{1}{2} \times AC \times BD\right)$
	**Trapezium** ① Perimeter $= a+b+c+d$ ② Area $= \frac{1}{2} \times$ (sum of // sides) $\times$ height $= \frac{1}{2} \times (b+d) \times h$
	**Rectangle** ① Perimeter $= 2(l+b)$ ② Area $=$ length $\times$ breadth $= l \times b$
	**Square** ① Perimeter $= 4l$ ② Area $= l \times l = l^2$
	**Triangle** ① Perimeter $= a+b+c$ ② Area $= \frac{1}{2} \times$ base $\times$ height $= \frac{1}{2} \times b \times h$ $\left(\text{or Area} = \frac{1}{2}ab\sin C\right)$

STOP AND THINK

[Q] The rectangle $PQRS$ measures 12 cm by 8 cm. Points T, U, V and W are on the sides with measurements, in centimetres, as shown. Find the area, in square centimetres, of $TUVW$. Find also the height, h, of the parallelogram $TUVW$.



SOLUTION

$PT = VR = 12 - 3 = 9$ cm
$QU = SW = 8 - 2 = 6$ cm

$$\begin{aligned}\text{Area of } TUVW &= \text{Area of } PQRS - (2\times\text{Area of }\Delta PWT) - (2\times\text{Area of }\Delta TQU)\\ &= (12\times 8) - \left(2\times\frac{1}{2}\times 2\times 9\right) - \left(2\times\frac{1}{2}\times 3\times 6\right)\\ &= 96 - 18 - 18\\ &= 60\text{ cm}^2 \quad \textbf{(Ans)}\end{aligned}$$

Apply Pythagoras' theorem, $WT = \sqrt{9^2 + 2^2} = \sqrt{85}$ cm

$$\text{Area of } TUVW = WT \times h$$

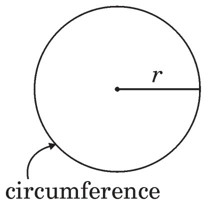
$$\Rightarrow \quad 60 = \sqrt{85} \times h$$

$$\therefore \; h = \frac{60}{\sqrt{85}}$$

$$\approx 6.51\text{ cm (3 sf)} \quad \textbf{(Ans)}$$

Circles

☑ We need the radius of a circle to find its ***circumference*** and ***area***:

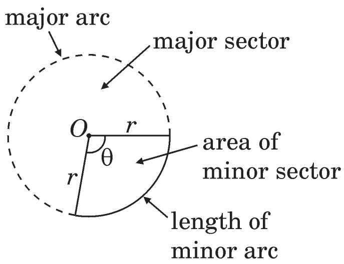


Circle

① Circumference $= 2\pi r$

② Area $= \pi r^2$ ← r = radius of circle
get value of π from calculators

☑ A portion of a circle bounded by two radii is called a sector. The angle of a sector is the angle included between the radii. To find ***arc length*** and ***area of sector***:



Sector of Circle

① Arc length of sector $= \dfrac{\theta}{360°} \times 2\pi r$

② Area of sector $= \dfrac{\theta}{360°} \times \pi r^2$

☑ If the angle of a sector is measured in radians (rad), the conversion is as follows:

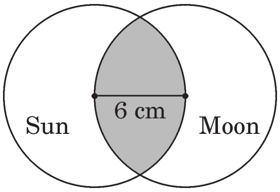
Degrees (°)	Radians (rad)
360°	2π
180°	π
1°	$\dfrac{\pi}{180}$

☑ The table shows the conversion from degrees to radians of some of the special angles most commonly used:

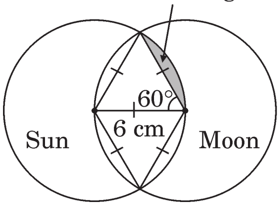
Angles in degrees (°)	30°	45°	60°	90°	180°	270°	360°
$\left(1° = \dfrac{\pi}{180}\text{ rad}\right)$	$30\times\dfrac{\pi}{180}$	$45\times\dfrac{\pi}{180}$	$60\times\dfrac{\pi}{180}$	$90\times\dfrac{\pi}{180}$	$180\times\dfrac{\pi}{180}$	$270\times\dfrac{\pi}{180}$	$360\times\dfrac{\pi}{180}$
Angles in radians (rad)	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$	π	$\dfrac{3}{2}\pi$	2π

STOP AND THINK

[Q] During an eclipse, the sun and the moon appear to be two equal circles of radius 6 cm, each passing through the centre of the other. Calculate the area of that part of the sun which is obscured.



SOLUTION



Area of one segment = Area of one sector − Area of one triangle

$$= \left(\frac{60^\circ}{360^\circ} \times \pi \times 6^2\right) - \left(\frac{1}{2} \times 6 \times 6 \times \sin 60^\circ\right)$$
$$= 6\pi - 18 \times \sin 60^\circ$$
$$= 3.26 \text{ cm}^2$$

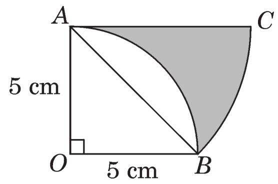
Area of the part where the sun is obscured

= Area of 2 triangles + Area of 4 segments

$$= \left(2 \times \frac{1}{2} \times 6 \times 6 \times \sin 60^\circ\right) + (4 \times 3.26)$$
$$= (36 \times \sin 60^\circ) + 13.04$$
$$= 44.22$$
$$\approx 44.2 \text{ cm}^2 \quad (3 \text{ sf})$$ **(Ans)**

STOP AND THINK

[Q]



The diagram, $\angle AOB = 90°$, $AC // OB$ and $OA = OB = 5$ cm. AB is an arc of a circle centre O and BC is an arc of circle centre A.

Find (i) the area of the shaded region,

(ii) the perimeter of the shaded region, giving your answer in terms of π and in surd form.

SOLUTION

(i) Area of segment AB = Area of sector OAB – Area of ΔOAB

$$= \left(\frac{90°}{360°} \times \pi \times 5^2\right) - \left(\frac{1}{2} \times 5 \times 5\right)$$
$$= \left(\frac{25\pi}{4} - \frac{25}{2}\right) \text{ cm}^2$$

Apply Pythagoras' theorem, $AC = AB \leftarrow$ radii of circle with centre A

$$= \sqrt{5^2 + 5^2}$$
$$= \sqrt{50} \text{ cm}$$

$\angle CAB = \angle OBA$ (alt. ∡s)

$= 45°$ (base ∡s in isosceles ΔOAB)

Area of sector $ABC = \dfrac{45°}{360°} \times \pi \times \left(\sqrt{50}\right)^2$

$$= \frac{1}{8} \times \pi \times 50$$
$$= \frac{25\pi}{4}$$

Area of shaded region $= \dfrac{25\pi}{4} - \left(\dfrac{25\pi}{4} - \dfrac{25}{2}\right)$

$$= \frac{25}{2}$$
$$= 12.5 \text{ cm}^2$$ **(Ans)**

(ii) Arc $AB = \dfrac{90°}{360°} \times 2 \times \pi \times 5$

$$= \frac{5\pi}{2} \text{ cm}$$

Arc $BC = \dfrac{45°}{360°} \times 2 \times \pi \times \sqrt{50}$

$$= \frac{\sqrt{50}}{4} \pi \text{ cm}$$

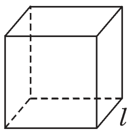
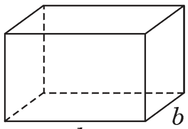
$\therefore$ Perimeter of shaded region $= \left(\dfrac{5\pi}{2} + \dfrac{\sqrt{50}\pi}{4} + \sqrt{50}\right)$ cm **(Ans)**

SURFACE AREA AND VOLUME OF SOLIDS

Objective: Find the surface area and volume of cube, cuboid, prism, cylinder, cone, pyramid, sphere and hemisphere.

Cube and cuboid

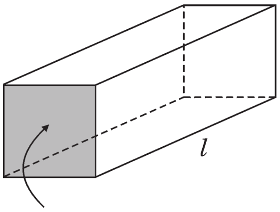
- ☑ All solids are 3-dimensional.
- ☑ A ***cube*** is a solid with equal length, breadth and height, so it has 6 common square surfaces.
- ☑ If the 3 sides are not the same, the solid is a ***cuboid***.
- ☑ To find ***total surface area*** and ***volume*** of cube and cuboid:

 	**Cube** ① Total surface area $= 6 \times \text{area of square}$ $= 6l^2$ ② Volume $= \text{length} \times \text{length} \times \text{length}$ $= l^3$
 	**Cuboid** ① Total surface area $= 2lb + 2lh + 2bh$ $= 2(lb + lh + bh)$ ② Volume $= \text{length} \times \text{breath} \times \text{height}$ $= lbh$

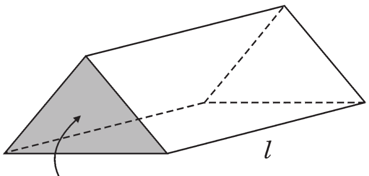
Prism

- ☑ Cube and cuboid are two types of prisms.
- ☑ A ***prism*** is a solid that has the same cross-section running through it.

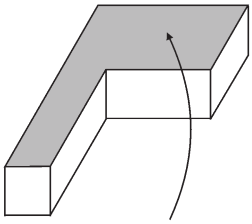
EXAMPLE



cross-sectional area, A



cross-sectional area, A



cross-sectional area, A

- ☑ To find ***volume*** and ***total surface area***:

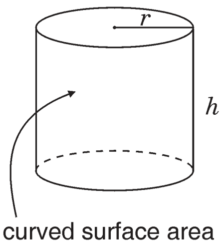
Prism

① Volume $= \text{area of cross-section} \times \text{length}$
$= Al$

② Total surface area $= (2 \times \text{area of cross-section})$
$+ (\text{perimeter of cross-section} \times \text{length})$

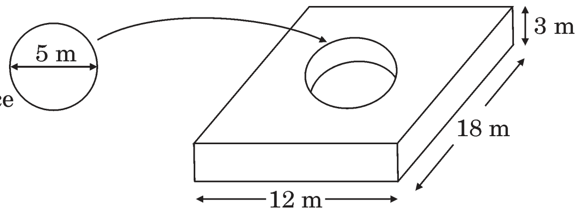
Cylinder

- ☑ A ***circular cylinder*** has a top and a bottom which are circles of equal size. Hence, it is a type of prisms with a circular cross-section.
- ☑ To find ***volume***, ***curved surface area*** and ***total surface area***:

	**Circular Cylinder** ① Volume = area of circle × height $= \pi r^2 h$ ② Curved surface area = perimeter of circle × height $= 2\pi rh$ ③ Total surface area = curved surface area + (2 × area of circle) $= 2\pi rh + 2\pi r^2$

STOP AND THINK

[Q] Find the volume and total surface area of the solid shown, taking $\pi = 3.14$.



SOLUTION

Volume of solid = volume of cuboid − volume of cylinder

$= (12 \times 18 \times 3) - (3.14 \times 2.5 \times 2.5 \times 3)$

$= 648 - 58.875$

$= 589.125$

$= 589 \text{ m}^3$ (3 sf) **(Ans)**

Total surface area ← $\boxed{\text{front \& back} + \text{left \& right} + \text{top \& bottom without circles} + \text{internal side wall of circular empty space}}$

$= 2(12 \times 3) + 2(18 \times 3) + 2\left[(12 \times 18) - (3.14 \times 2.5 \times 2.5)\right] + (2 \times 3.14 \times 2.5 \times 3)$

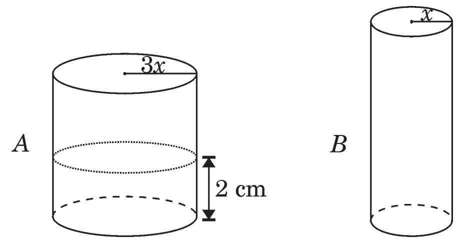
$= 72 + 108 + 392.75 + 47.1$

$= 619.85$

$= 620 \text{ m}^2$ (3 sf) **(Ans)**

STOP AND THINK

[Q] Two cylindrical jars A and B have radii of $3x$ centimetres and x centimetres respectively. Initially, cylinder B is empty and cylinder A contains water to a depth of 2 cm. If this water is all poured into cylinder B, calculate the height it will reach.



SOLUTION

Let height of water in cylinder B be h.

$$\text{Volume of water in cylinder } A = \pi \times (3x)^2 \times 2 = 18\pi x^2 \text{ cm}^3$$

$$\text{Volume of water from } A \text{ in cylinder } B = \pi x^2 h$$

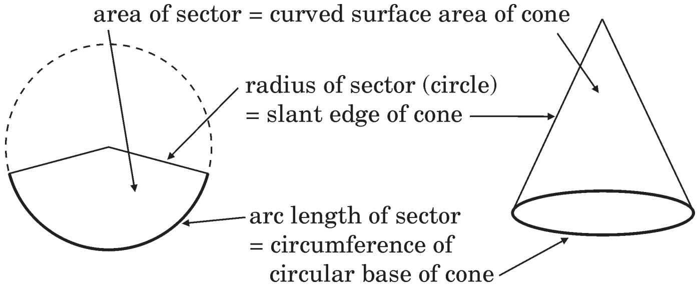
$$\Rightarrow \quad 18\pi x^2 = \pi x^2 h$$

$$\therefore \quad h = \frac{18\pi x^2}{\pi x^2} = 18 \text{ cm} \quad \textbf{(Ans)}$$

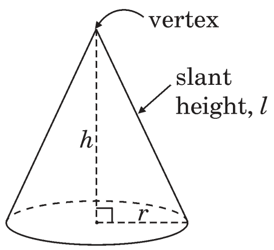
Cone

☑ A ***circular cone*** is a solid with a circular base and a vertex.

☑ The curved surface is formed from sector of a circle.

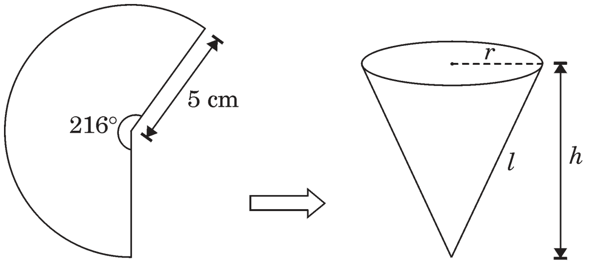


☑ To find ***volume***, ***curved surface area*** and ***total surface area***:

	**Circular Cone** ① Volume $= \frac{1}{3}\pi r^2 h$ ② Curved surface area $= \pi r l$ ③ Total surface area $=$ curved surface area $+$ area of circle $= \pi r l + \pi r^2$

STOP AND THINK

[Q] A piece of paper in the form of a circular sector is made to form a right circular cone. The radius of the sector is 5 cm. The angle at the centre is 216°.



sector of a circle right circular cone

(i) Write down the length of the slant edge l of the cone formed.

(ii) Find the radius r of the circular base of the cone.

(iii) The curved surface area of the cone can be obtained using the formula $\pi r l$ where r is the base radius and l the slant height of the cone. Find the curved surface area leaving your answer in terms of π.
Suggest another way of finding the curved surface area and show that the same answer is obtained using the suggested method.

(iv) Taking π to be 3.142, find the capacity of the cone.

SOLUTION

(i) Slant edge of cone = radius of sector

$$\therefore \ l = 5 \text{ cm} \quad \textbf{(Ans)}$$

(ii) $$\text{Arc length of sector} = \frac{216°}{360°} \times 2 \times \pi \times 5 = 6\pi \text{ cm}$$

Circumference of circular base of cone = arc length of sector

$$\Rightarrow \quad 2\pi r = 6\pi$$
$$\therefore \ r = 3 \text{ cm} \quad \textbf{(Ans)}$$

(iii) $$\text{Curved surfaced area of cone} = \pi \times 3 \times 5 = 15\pi \text{ cm}^2 \quad \textbf{(Ans)}$$

Another way of finding curved surface area is to find the area of the sector.

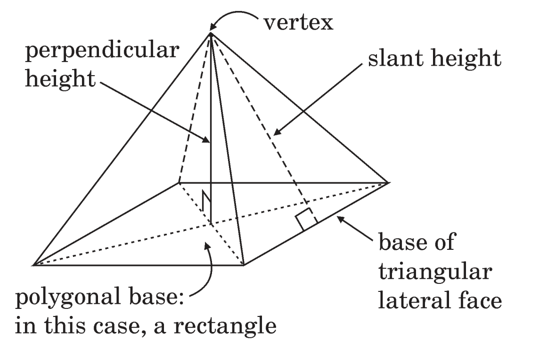
$$\text{Area of sector} = \frac{216°}{360°} \times \pi \times 5 \times 5 = 15\pi \text{ cm}^2 \quad \text{(shown)}$$

(iv) Apply Pythagoras' theorem, height of cone, $h = \sqrt{5^2 - 3^2} = 4$ cm

$$\begin{aligned}\text{Capacity of cone} &= \text{volume of cone}\\ &= \frac{1}{3} \times \pi \times 3^2 \times 4\\ &= 3.142 \times 12\\ &= 37.704\\ &= 37.7 \text{ cm}^3 \ \text{(3 sf)} \quad \textbf{(Ans)}\end{aligned}$$

Pyramid

- ☑ A ***regular pyramid*** is a solid that has a regular polygonal base with a perpendicular vertex and slant triangular lateral faces.
- ☑ The polygonal base can be a triangle, a square, a rectangle, etc.
- ☑ To find ***volume*** and ***total surface area***:



Regular Pyramid

① Volume $= \frac{1}{3} \times$ base area $\times$ height

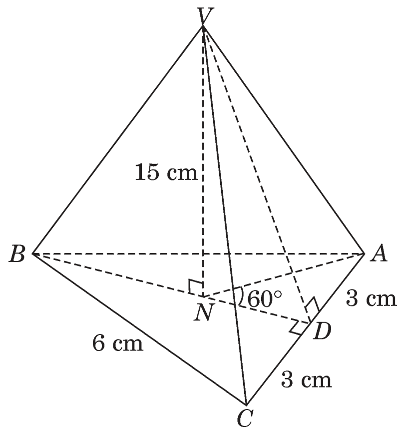
② Total surface area
$=$ base area
$+$ total area of slant Δ faces

area of slant Δ face $= \frac{1}{2} \times$ base $\times$ slant height

STOP AND THINK

[Q] The base ABC of the pyramid is an equilateral triangle whose sides are 6 cm long.

Given that $\angle AND = 60°$ and the height, VN, of the pyramid is 15 cm, find the volume of the pyramid and total area of the slant triangular faces of the pyramid.



SOLUTION

$$\text{Volume of pyramid } = \frac{1}{3} \times \left(\frac{1}{2} \times 6 \times 6 \times \sin 60° \right) \times 15$$
$$= 77.94$$
$$\approx 77.9 \text{ cm}^3 \quad (3 \text{ sf}) \quad \textbf{(Ans)}$$

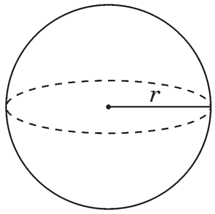
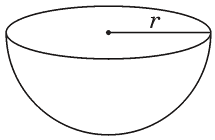
$$ND = \frac{3}{\tan 60°} = \frac{3}{\sqrt{3}} = \sqrt{3} \text{ cm}$$

$$VD = \sqrt{15^2 + ND^2} = \sqrt{225 + 3} = \sqrt{228} \text{ cm}$$

$$\text{Total area of slant triangular faces } = 3\left(\frac{1}{2} \times 6 \times \sqrt{228} \right)$$
$$= 135.89$$
$$\approx 136 \text{ cm}^2 \quad (3 \text{ sf}) \quad \textbf{(Ans)}$$

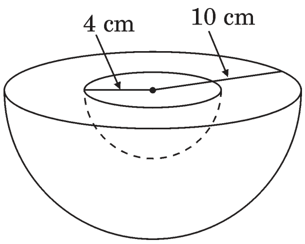
Sphere and hemisphere

- ☑ Every point on the surface of a ***sphere*** is equidistance from its centre.
- ☑ A ***hemisphere*** is half a sphere.
- ☑ To find ***surface area*** and ***volume***:

	**Sphere** ① Spherical surface area $= 4\pi r^2$ ② Volume $= \frac{4}{3}\pi r^3$
	**Hemisphere** ① Total surface area $= \frac{1}{2} \times$ spherical surface area + area of circle $= 2\pi r^2 + \pi r^2 = 3\pi r^2$ ② Volume $= \frac{1}{2} \times$ volume of sphere $= \frac{2}{3}\pi r^3$

STOP AND THINK

[Q] The diagram shows a hemispherical solid with a hollow centre. Their radii are 10 cm and 4 cm respectively. Find the volume and surface area of the hemispherical solid leaving your answer in terms of π.



SOLUTION

$$\begin{array}{l}\text{Volume of}\\ \text{hemispherical solid}\end{array} = \begin{array}{l}\text{volume of}\\ \text{outer hemishpere}\end{array} - \begin{array}{l}\text{volume of}\\ \text{inner hemishpere}\end{array}$$

$$= \frac{2}{3} \times \pi \times \left(10^3 - 4^3\right)$$

$$= 624\pi \text{ cm}^3 \quad \textbf{(Ans)}$$

$$\begin{array}{l}\text{Surface area of}\\ \text{hemispherical solid}\end{array} = \begin{array}{l}\text{outer spherical}\\ \text{surface area}\end{array} + \begin{array}{l}\text{surface area of}\\ \text{spherical inner wall}\end{array} + \begin{array}{l}\text{area of}\\ \text{circular base}\end{array}$$

$$= \left(2\pi \times 10^2\right) + \left(2\pi \times 4^2\right) + \pi\left(10^2 - 4^2\right)$$

$$= 200\pi + 32\pi + 84\pi$$

$$= 316\pi \text{ cm}^2 \quad \textbf{(Ans)}$$

TOPIC 15

STATISTICS

OVERVIEW OF THE TOPIC

STATISTICS *5 key areas*	DATA REPRESENTATION	• Frequency tables • Bar charts • Pie charts • Line graphs • Pictograms • Dot diagrams • Stem and leaf diagrams • Histograms
	MEASURES OF CENTRAL TENDENCY	• Mean • Median • Mode
	STANDARD DEVIATION	
	CUMULATIVE FREQUENCY	
	THE BOX-AND-WHISKER PLOTS	

DATA REPRESENTATION

Objective: Represent data in different forms of graphs.

Frequency tables

☑ A ***frequency table*** is a an easy way of organising collected data.

EXAMPLE A Frequency Table

Number Of Computers At Home	Frequency (Number Of Students Surveyed)
0	5
1	30
2	25
3 or more	10

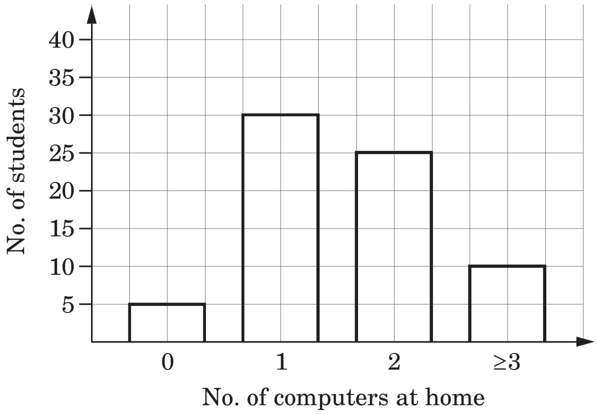
Bar charts

☑ ***Bar charts*** are easy to construct and understand.

☑ The bars are equal in width (should be evenly spaced out) and can be drawn vertically or horizontally.

☑ The height (or length) of each bar is proportional to the data represented.

EXAMPLE A Bar Chart



STOP AND THINK

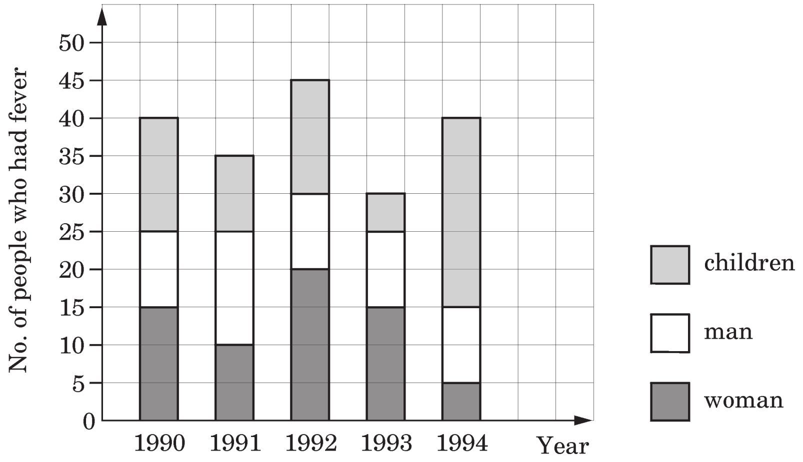
[Q] The bar chart shows the number of people who had fever over the years. The people are classified into three categories, namely, children, man and woman.

(a) Which category has a constant number over three consecutive years? State the number and the years.

(b) In which year does the woman category account for the highest percentage of people who had fever?

(c) What is the percentage increase in the number of children who had fever from 1993 to 1994?

(d) What are the years when the total number of man and children who had fever is the same?



SOLUTION

(a) Man category: 10 men each in year 1992, year 1993 and year 1994. **(Ans)**

(b) Year 1993. **(Ans)** ←

1993	Note: not 1992
$\frac{15}{30} \times 100\% = 50\%$	$\frac{20}{45} \times 100\% = 44.4\%$

(c) From 5 children to 25 children: $25 - 5 = 20$

Percentage increase: $\frac{20}{5} \times 100\% = 400\%$ **(Ans)**

(d) Year 1990, year 1991 and year 1992. **(Ans)**

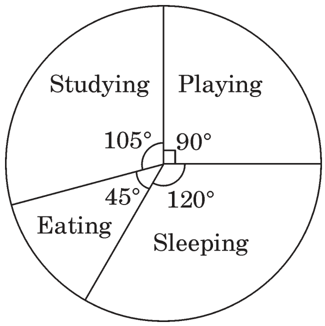
Pie charts

- ☑ A ***pie chart*** is a circle divided into sectors whose angles are proportional to the data represented.
- ☑ Pie charts are useful when the number of classes is small and when we want to show the proportions rather than the frequencies.

EXAMPLE

A boy spends each day eating, studying, playing and sleeping. The hours spent on each of these activities are given in the table below.

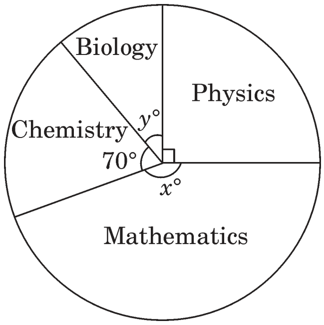
Activity	Hours spent	Angle of sector
Eating	3	$\frac{3}{24} \times 360° = 45°$
Studying	7	$\frac{7}{24} \times 360° = 105°$
Playing	6	$\frac{6}{24} \times 360° = 90°$
Sleeping	8	$\frac{8}{24} \times 360° = 120°$
Total	24	360°



A Pie Chart

STOP AND THINK

[Q] The pie chart shows the number of students major in various subjects in a certain college. There are altogether 500 students major in Biology and Mathematics.



(a) Find the number of students altogether.

(b) If there are altogether 275 students major in Chemistry and Biology, find the values of x and y.

SOLUTION

(a)
$$x + y = 360° - 90° - 70°$$
$$= 200°$$
$$200° \rightarrow 500 \text{ students}$$
$$1° \rightarrow \left(\frac{500}{200}\right) \text{ students}$$
$$= \frac{5}{2} \text{ students}$$
$$\therefore\ 360° \rightarrow \left(\frac{5}{2} \times 360\right) \text{ students}$$
$$= 900 \text{ students} \quad \textbf{(Ans)}$$

(b) Number of students major in Chemistry $= \frac{70°}{360°} \times 900$
$= 175$ students

Number of students major in Biology $= 275 - 175$
$= 100$ students

$$\therefore\ y = \frac{100}{900} \times 360°$$
$$= 40° \quad \textbf{(Ans)}$$

$$x + y = 200°$$
$$\Rightarrow \quad x + 40° = 200°$$
$$\therefore\ x = 160° \quad \textbf{(Ans)}$$

Line graphs

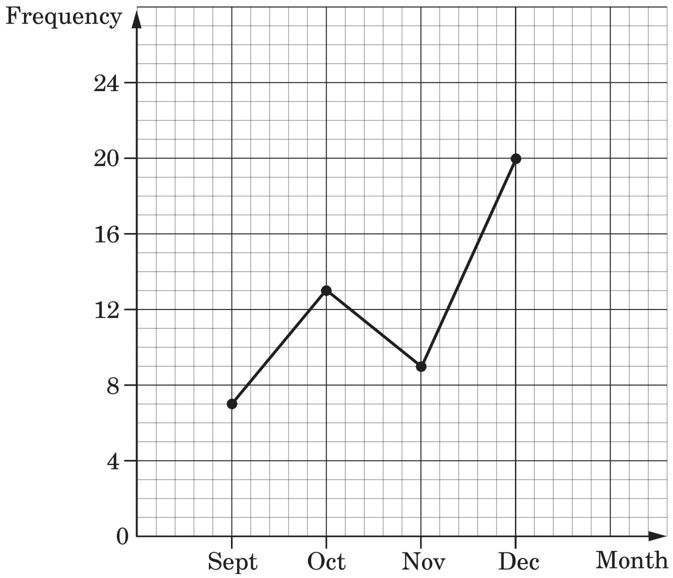
☑ A ***line graph*** is drawn by plotting the points corresponding to the data and then joining the points by line segments.

EXAMPLE

The table below shows the number of houses sold by an agent over a period of 4 months.

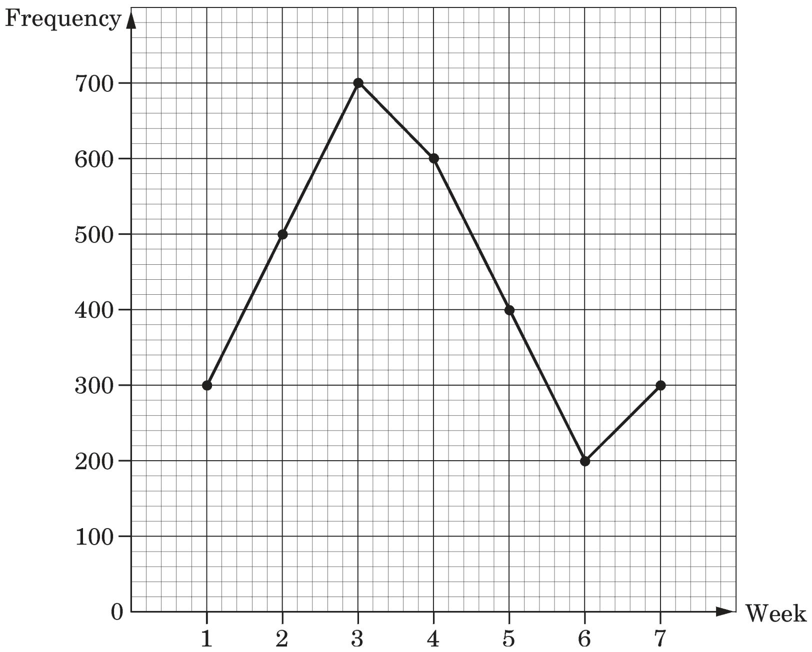
Month	Frequency
Sept	7
Oct	13
Nov	9
Dec	20

A Line Graph



STOP AND THINK

[Q] The weekly sales of singles in a record shop are displayed on the following graph.



(a) Write down the number of singles sold in each of the seven weeks.

(b) Describe in your own words the pattern of sales during these seven weeks.

SOLUTION

(a) week 1 = 300, week 2 = 500, week 3 = 700, week 4 = 600,

week 5 = 400, week 6 = 200, week 7 = 300 **(Ans)**

(b) It increases from week 1 to week 3, then it declines in sales all the way to week 6 before it increases again in week 7. **(Ans)**

Pictograms

☑ In a ***pictogram***, a symbol or an icon is used to represent a quantity of items.

EXAMPLE

A group of students was asked to choose their favourite holiday destination.

A Pictogram

Each ☺ represents 18 students

Dot diagrams *(not in the syllabus)*

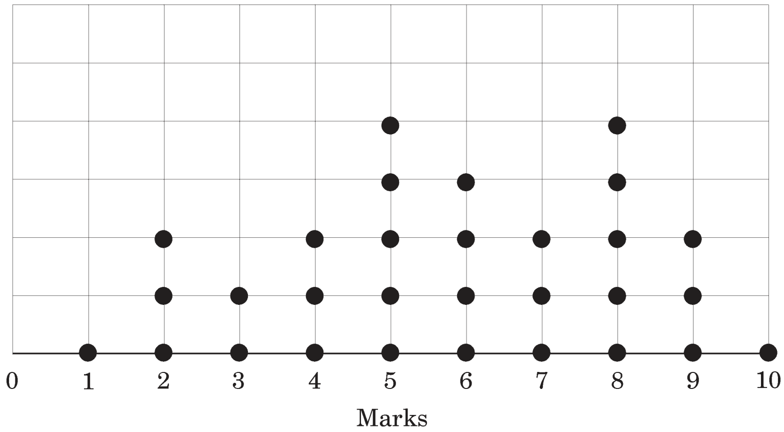
☑ A ***dot diagram*** is drawn by placing dots that represent the values of a set of data above a horizontal line.

☑ The number of dots above each value indicates how many times each value occured.

EXAMPLE The marks of 30 pupils are given below.

8	3	8	9	4	9	5	7	6	2
1	4	5	8	7	6	6	2	3	5
10	8	2	7	6	9	5	8	4	5

A Dot Diagram



Stem and leaf diagrams *(not in the syllabus)*

- ☑ In a ***stem and leaf diagram***, each value is split into two parts, the stem and the leaf by a vertical line.
- ☑ The stems represent the ten digits and the leaves represent the ones digit.

EXAMPLE

The mass of 20 boys are given below.

64	40	58	60	39	45	38	53	65	65
38	39	57	40	54	58	60	64	53	45

A Stem & Leaf Diagram

S	L
3	8 8 9 9
4	0 0 5 5
5	3 3 4 7 8 8
6	0 0 4 4 5 5

STOP AND THINK

[Q] The weights, in kilograms, of a group of 15 students in a class are represented in the stem and leaf diagram below.

3	9 9
4	0 1 1 5 6
5	2 4 6 7 8
6	0 4
7	3

(a) Find the weights of the lightest and heaviest student in the above group.

(b) The weights, in kilograms, of the rest of the 20 students in the class are shown below.

71 56 56 40 67 75 41 68 60 49

68 73 73 56 64 45 80 38 82 59

Construct a single ordered stem and leaf diagram to represent the weights of all the 35 students in the class.

(c) For the whole class, find

(i) the most common weight,

(ii) the weight of the heaviest student,

(iii) the percentage of the students who weighed at least 60 kg.

(d) Construct a dot diagram to represent the above data.

SOLUTION

(a) Weight of the lightest student = 39 kg **(Ans)**

Weight of the heaviest student = 73 kg **(Ans)**

(b)

3	8 9 9
4	0 0 1 1 1 5 5 6 9
5	2 4 6 6 6 6 7 8 9
6	0 0 4 4 7 8 8
7	1 3 3 3 5
8	0 2

(Ans)

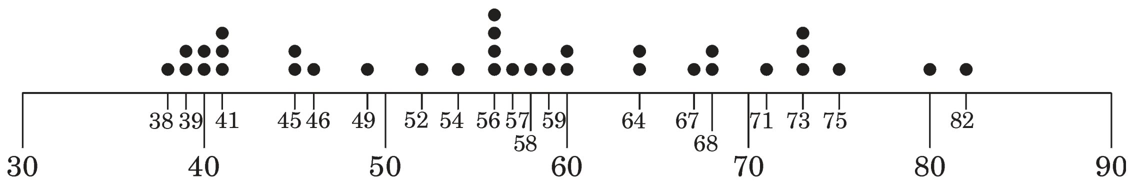
(c) (i) The most common weight = 56 kg **(Ans)**

(ii) Weight of heaviest student = 82 kg **(Ans)**

(iii) % of students who weighed at least 60 kg $= \frac{14}{35} \times 100\%$

$= 40\%$ **(Ans)**

(d)



Histograms

☑ A ***histogram*** is a vertical bar chart with NO gaps between the bars. The base of each bar represents a class interval of a set of grouped data.

☑ THE FIRST TYPE OF HISTOGRAMS: Grouped data with equal class intervals. In this case, the heights of the bar is proportional to the class frequencies.

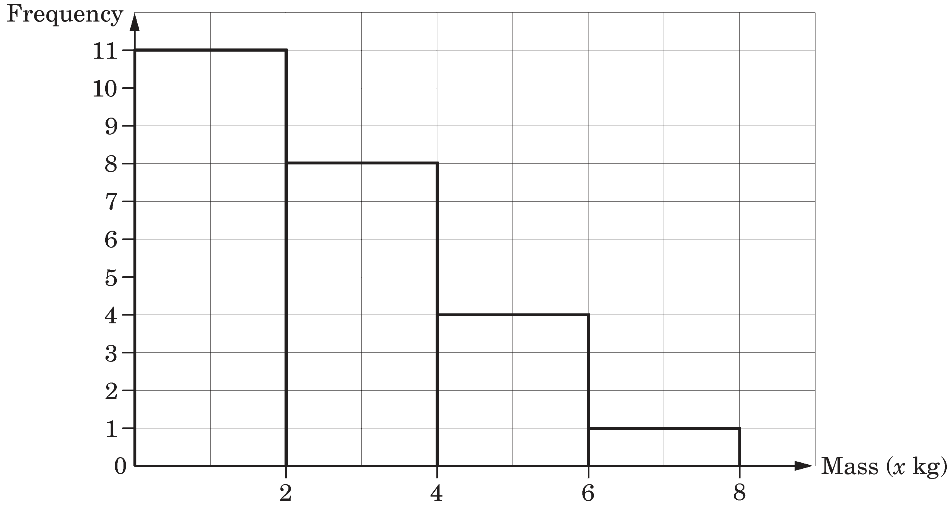
EXAMPLE

The mass, to the nearest kilogram, of 24 fruits are shown in the table below.

grouped data

equal class intervals

Mass (x kg)	Frequency
$0 < x \le 2$ kg	11
$2 < x \le 4$ kg	8
$4 < x \le 6$ kg	4
$6 < x \le 8$ kg	1



A Histogram (equal class intervals)

☑ THE SECOND TYPE OF HISTOGRAMS: Grouped data with unequal class intervals. In this case, the <u>areas</u> of the bars are proportional to the class frequencies.

☑ To find heights, apply equation: $\boxed{\text{Frequency density} = \dfrac{\text{Class frequency}}{\text{Class width}}}$

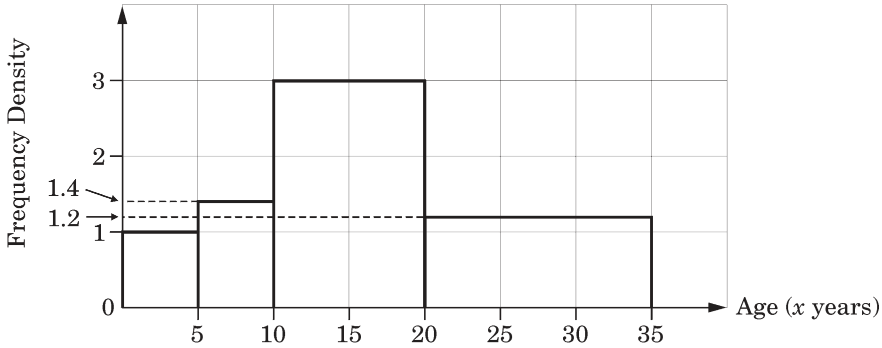
EXAMPLE

The histogram shows the ages, in years, of 60 people in a zoo.

Age (x years)	Frequency
$0 < x \le 5$	5
$5 < x \le 10$	7
$10 < x \le 20$	30
$20 < x \le 35$	18

$\Rightarrow$

Class Width	Frequency Density
5	$5 \div 5 = 1$
5	$7 \div 5 = 1.4$
10	$30 \div 10 = 3$
15	$18 \div 15 = 1.2$



A Histogram (unequal class intervals)

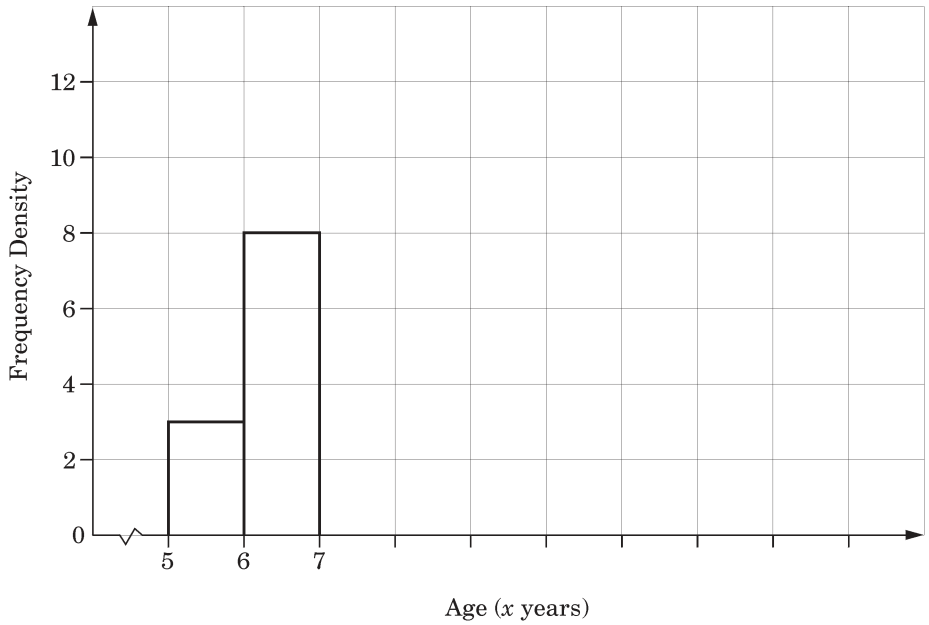
STOP AND THINK

[Q] The following table shows the grouped frequency distribution of the ages of a group of children.

Age (x years)	$5 < x \le 6$	$6 < x \le 7$	$7 < x \le 9$	$9 < x \le 10$	$10 < x \le 14$
Number of children	3	8	12	7	6

(a) Find the percentage of the group whose age is more than 6 and less than or equal to 10.

(b) Complete the histogram which represents this information.



SOLUTION

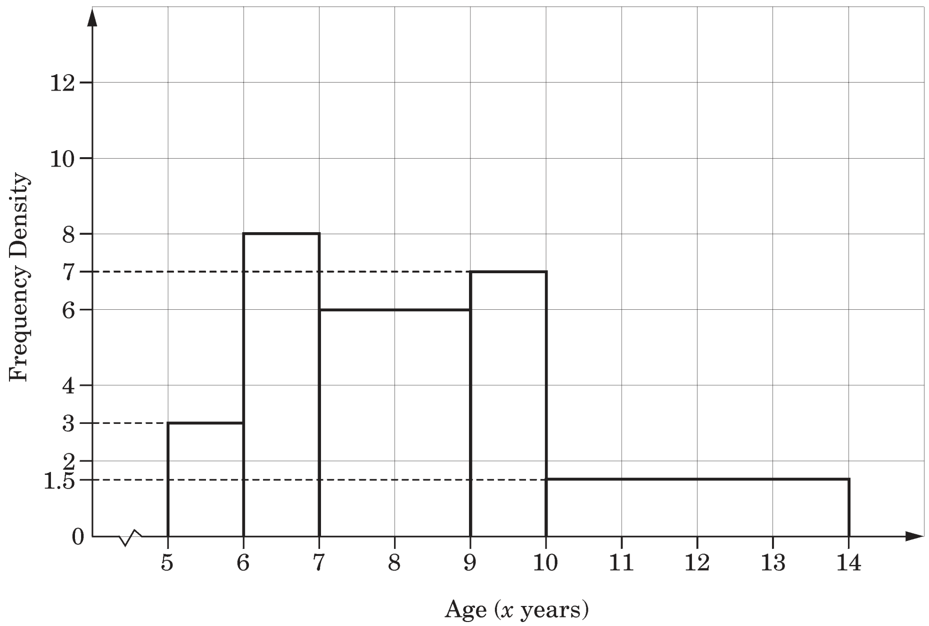
(a) Number of children in the group whose age is $6 < x \leq 10$ $= 8 + 12 + 7$
$= 27$

Total number of children = 3 + 8 + 12 + 7 + 6 = 36

Percentage $= \dfrac{27}{36} \times 100\%$
$= 75\%$ **(Ans)**

(b)

Class width	1	1	2	1	4
Number of children	3	8	12	7	6
Frequency density	3	8	6	7	1.5



MEASURES OF CENTRAL TENDENCY

Objectives: Compute mean, median and mode for grouped and ungrouped data and to use these measures in comparing two sets of data.

Mean

☑ ***Mean*** is the average value of a set of values. It is obtained by dividing the sum of all the values by the number of items in the set.

☑

Mean For Ungrouped Data	For a set of numbers $\{x_1, x_2, x_3, \dots x_n\}$, mean, $\bar{x} = \dfrac{x_1 + x_2 + x_3 + \dots + x_n}{n}$	n = number of items in the set

☑

Mean For Grouped Data	mean, $\bar{x} = \dfrac{\sum fx}{\sum f}$	where $\sum$ = the sum of x = mid-value of the class interval f = frequency of the class interval

Median

☑ The ***median*** is the value of the middle term when a set of data is arranged in ascending order.

☑ If there are n terms and if

❶ n is odd, then the median is the $\left(\dfrac{n+1}{2}\right)^{\text{th}}$ term.

❷ n is even, then the median is the average of the two middle terms.

Mode

☑ The ***mode*** is the number which occurs the most often in a set of values.

☑ There may be more than 1 mode for some distribution or even no mode.

☑ For grouped data, the class interval with the highest frequency is called the modal class.

STOP AND THINK

[Q] For the numbers 3, 4, 8, –4, 7, –5, 6, 4, 9, 4, find
(a) the mean,
(b) the median,
(c) the mode.

SOLUTION

(a) $$\text{Mean} = \frac{3+4+8+(-4)+7+(-5)+6+4+9+4}{10}$$
$$= \frac{36}{10}$$
$$= 3.6 \quad \textbf{(Ans)}$$

(b) In ascending order: $-5,\ -4,\ 3,\ 4,\ \underbrace{4,\ 4,}_{\text{two middle terms}}\ 6,\ 7,\ 8,\ 9$

$$\text{Median} = \frac{4+4}{2}$$
$$= 4 \quad \textbf{(Ans)}$$

(c) Mode = 4 **(Ans)**

STOP AND THINK

[Q] In a random survey of 100 adults, their shoe sizes were noted and tabulated as follows:

Shoe size	6	7	8	9	10
Number of adults	10	10	20	40	20

(a) Write down the modal shoe size.
(b) Find the median shoe size.
(c) Calculate the mean shoe size.

SOLUTION

(a) 40 is the highest number of occurence.

Modal shoe size = 9 **(Ans)**

(b) Total number of adults is 100 which is even. The middle two terms 50th and 51st are both shoe size 9.

Median shoe size = 9 **(Ans)**

(c) $$\text{Mean shoe size} = \frac{\sum fx}{\sum f}$$
$$= \frac{6\times10+7\times10+8\times20+9\times40+10\times20}{100}$$
$$= \frac{60+70+160+360+200}{100}$$
$$= \frac{850}{100}$$
$$= 8.5 \quad \textbf{(Ans)}$$